1 Complex numbers

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

Cartesian form: $a + bi$

Polar form: $r \operatorname{cis} \theta$

Operations

	Cartesian	**Polar**
$z_1 \pm z_2$	$(a \pm c)(b \pm d)i$	convert to $a + bi$
$+k \times z$	$ka \pm kbi$	$kr \operatorname{cis} \theta$
$-k \times z$		$kr \operatorname{cis}(\theta \pm \pi)$
$z_1 \cdot z_2$	$ac - bd + (ad + bc)i$	$r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$
$z_1 \div z_2$	$(z_1 \overline{z_2}) \div \|z_2\|^2$	$\left(\frac{r_1}{r_2}\right) \operatorname{cis}(\theta_1 - \theta_2)$

Scalar multiplication in polar form

For $k \in \mathbb{R}^+$:

$$k\,(r \operatorname{cis} \theta) = kr \operatorname{cis} \theta$$

For $k \in \mathbb{R}^-$:

$$k\,(r \operatorname{cis} \theta) = kr \operatorname{cis} \left(\begin{cases} \theta - \pi & 0 < \operatorname{Arg}(z) \le \pi \\ \theta + \pi & -\pi < \operatorname{Arg}(z) \le 0 \end{cases} \right)$$

Conjugate

$$\begin{aligned} \overline{z} &= a \mp bi \\ &= r \operatorname{cis}(-\theta) \end{aligned}$$

On CAS: `conjg(a+bi)`

Properties

$$\begin{aligned} \overline{z_1 \pm z_2} &= \overline{z_1} \pm \overline{z_2} \\ \overline{z_1 \cdot z_2} &= \overline{z_1} \cdot \overline{z_2} \\ \overline{kz} &= k\overline{z} \quad | \quad k \in \mathbb{R} \\ z\overline{z} &= (a + bi)(a - bi) \\ &= a^2 + b^2 \\ &= |z|^2 \end{aligned}$$

Modulus

$$|z| = |\vec{Oz}| = \sqrt{a^2 + b^2}$$

Properties

$$\begin{aligned} |z_1 z_2| &= |z_1||z_2| \\ \left|\frac{z_1}{z_2}\right| &= \frac{|z_1|}{|z_2|} \\ |z_1 + z_2| &\le |z_1| + |z_2| \end{aligned}$$

Multiplicative inverse

$$\begin{aligned} z^{-1} &= \frac{a - bi}{a^2 + b^2} \\ &= \frac{\overline{z}}{|z|^2} a \\ &= r \operatorname{cis}(-\theta) \end{aligned}$$

Dividing over $\mathbb{C}$

$$\begin{aligned} \frac{z_1}{z_2} &= z_1 z_2^{-1} \\ &= \frac{z_1 \overline{z_2}}{|z_2|^2} \\ &= \frac{(a + bi)(c - di)}{c^2 + d^2} \end{aligned}$$

(rationalise denominator)

Polar form

$$\begin{aligned} z &= r \operatorname{cis} \theta \\ &= r(\cos \theta + i \sin \theta) \end{aligned}$$

- $r = |z| = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}$
- $\theta = \arg(z)$ On CAS: `arg(a+bi)`
- $\operatorname{Arg}(z) \in (-\pi, \pi)$ **(principal argument)**
- Convert on CAS:
  `compToTrig(a+bi)` $\iff$ `cExpand{r·cisX}`
- Multiple representations:
  $r \operatorname{cis} \theta = r \operatorname{cis}(\theta + 2n\pi)$ with $n \in \mathbb{Z}$ revolutions
- $\operatorname{cis} \pi = -1, \qquad \operatorname{cis} 0 = 1$

de Moivres' theorem

$$(r \operatorname{cis} \theta)^n = r^n \operatorname{cis}(n\theta) \text{ where } n \in \mathbb{Z}$$

Complex polynomials

Include $\pm$ for all solutions, incl. imaginary

Sum of squares	$z^2 + a^2 = z^2 - (ai)^2$ $= (z + ai)(z - ai)$
Sum of cubes	$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
Division	$P(z) = D(z)Q(z) + R(z)$
Remainder theorem	Let $\alpha \in \mathbb{C}$. Remainder of $P(z) \div (z - \alpha)$ is $P(\alpha)$
Factor theorem	$z - \alpha$ is a factor of $P(z) \iff P(\alpha) = 0$ for $\alpha \in \mathbb{C}$
Conjugate root theorem	$P(z) = 0$ at $z = a \pm bi$ ($\implies$ both z_1 and $\overline{z_1}$ are solutions)

Roots

nth roots of $z = r \operatorname{cis} \theta$ are:

$$z = r^{\frac{1}{n}} \operatorname{cis}\left(\frac{\theta + 2k\pi}{n}\right)$$

- Same modulus for all solutions
- Arguments are separated by $\frac{2\pi}{n}$
- Solutions of $z^n = a$ where $a \in \mathbb{C}$ lie on the circle $x^2 + y^2 = \left(|a|^{\frac{1}{n}}\right)^2$ (intervals of $\frac{2\pi}{n}$)

For $0 = az^2 + bz + c$, use quadratic formula:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

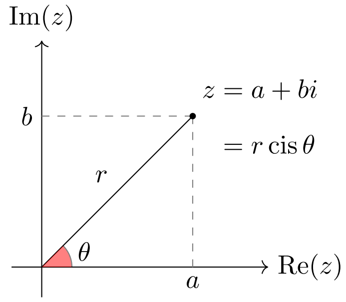
Fundamental theorem of algebra

A polynomial of degree n can be factorised into n linear factors in $\mathbb{C}$:

$$\implies P(z) = a_n(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)\dots(z - \alpha_n)$$

where $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n \in \mathbb{C}$

Argand planes



- Multiplication by $i \implies$ CCW rotation of $\frac{\pi}{2}$
- Addition: $z_1 + z_2 \equiv \overrightarrow{Oz_1} + \overrightarrow{Oz_2}$

Sketching complex graphs

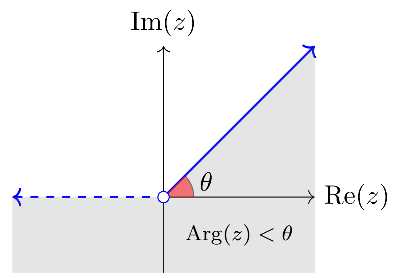
Linear

- $\text{Re}(z) = c$ or $\text{Im}(z) = c$ (perpendicular bisector)
- $\text{Im}(z) = m\,\text{Re}(z)$
- $|z + a| = |z + b| \implies 2(a - b)x = b^2 - a^2$

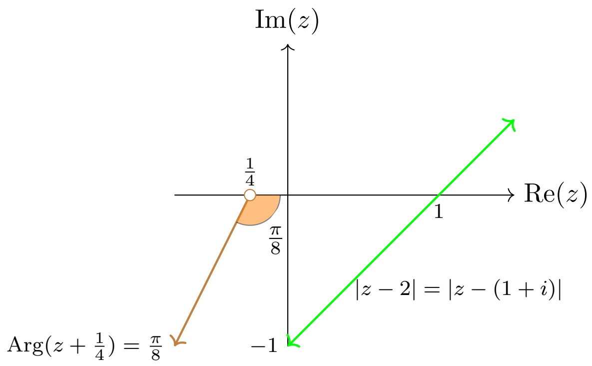
Circles

- $|z - z_1|^2 = c^2|z_2 + 2|^2$
- $|z - (a + bi)| = c$

Loci $\text{Arg}(z) < \theta$



Rays $\text{Arg}(z - b) = \theta$



2 Vectors

- **vector:** a directed line segment
- arrow indicates direction
- length indicates magnitude
- notated as $\vec{a}, \widetilde{A}, \vec{a}$
- column notation: $\begin{bmatrix} x \\ y \end{bmatrix}$
- vectors with equal magnitude and direction are equivalent

2.1 Vector addition

$\boldsymbol{u} + \boldsymbol{v}$ can be represented by drawing each vector head to tail then joining the lines.
Addition is commutative (parallelogram)

2.2 Scalar multiplication

For $k \in \mathbb{R}^+$, $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$ but length is multiplied by a factor of k.

When multiplied by $k < 0$, direction is reversed and length is multplied by k.

2.3 Vector subtraction

To find $\boldsymbol{u} - \boldsymbol{v}$, add $-\boldsymbol{v}$ to $\boldsymbol{u}$

2.4 Parallel vectors

Same or opposite direction

$$\boldsymbol{u} || \boldsymbol{v} \iff \boldsymbol{u} = k\boldsymbol{v} \text{ where } k \in \mathbb{R} \setminus \{0\}$$

2.5 Position vectors

Vectors may describe a position relative to O.

For a point A, the position vector is $\overrightarrow{OA}$

2.6 Linear combinations of non-parallel vectors

If two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel, then:

$$m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b} \quad \therefore \quad m = p,\ n = q$$

2.7 Column vector notation

A vector between points $A(x_1, y_1)$, $B(x_2, y_2)$ can be represented as $\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$

2.8 Component notation

A vector $\boldsymbol{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ can be written as $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$.
$\boldsymbol{u}$ is the sum of two components $x\boldsymbol{i}$ and $y\boldsymbol{j}$
Magnitude of vector $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$ is denoted by $|u| = \sqrt{x^2 + y^2}$

Basic algebra applies:
$(x\boldsymbol{i} + y\boldsymbol{j}) + (m\boldsymbol{i} + n\boldsymbol{j}) = (x + m)\boldsymbol{i} + (y + n)\boldsymbol{j}$
Two vectors equal if and only if their components are equal.

2.9 Unit vector $|\hat{\boldsymbol{a}}| = 1$

$$\begin{aligned} \hat{\boldsymbol{a}} &= \frac{1}{|\boldsymbol{a}|}\boldsymbol{a} \\ &= \boldsymbol{a} \cdot |\boldsymbol{a}| \end{aligned} \tag{1}$$

Scalar/dot product $\boldsymbol{a} \cdot \boldsymbol{b}$

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$

on CAS: `dotP([a b c], [d e f])`

2.10 Scalar product properties

1. $k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (k\boldsymbol{b})$
2. $\boldsymbol{a} \cdot \boldsymbol{0} = 0$
3. $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
4. $\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1$
5. If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular
6. $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2 = a^2$

For parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{a}\cdot\boldsymbol{b} = \begin{cases} |\boldsymbol{a}||\boldsymbol{b}| & \text{if same direction} \\ -|\boldsymbol{a}||\boldsymbol{b}| & \text{if opposite directions} \end{cases}$$

2.11 Geometric scalar products

$$\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$$

where $0 \le \theta \le \pi$

2.12 Perpendicular vectors

If $\boldsymbol{a}\cdot\boldsymbol{b} = 0$, then $\boldsymbol{a} \perp \boldsymbol{b}$ (since $\cos 90 = 0$)

2.13 Finding angle between vectors

positive direction

$$\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{a_1b_1 + a_2b_2}{|\boldsymbol{a}||\boldsymbol{b}|}$$

on CAS: `angle([a b c], [a b c])` (Action -> Vector -> Angle)

2.14 Angle between vector and axis

Direction of a vector can be given by the angles it makes with $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ directions.

For $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ which makes angles α, β, γ with positive direction of x, y, z axes:

$$\cos\alpha = \frac{a_1}{|\boldsymbol{a}|}, \quad \cos\beta = \frac{a_2}{|\boldsymbol{a}|}, \quad \cos\gamma = \frac{a_3}{|\boldsymbol{a}|}$$

on CAS: `angle([a b c], [1 0 0])` for angle between $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ and x-axis

2.15 Vector projections

Vector resolute of $\boldsymbol{a}$ in direction of $\boldsymbol{b}$ is magnitude of $\boldsymbol{a}$ in direction of $\boldsymbol{b}$:

$$\boldsymbol{u} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|^2}\boldsymbol{b} = \left(\boldsymbol{a}\cdot\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right)\left(\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right) = (\boldsymbol{a}\cdot\hat{\boldsymbol{b}})\hat{\boldsymbol{b}}$$

2.16 Scalar resolute of $\boldsymbol{a}$ on $\boldsymbol{b}$

$$r_s = |\boldsymbol{u}| = \boldsymbol{a}\cdot\hat{\boldsymbol{b}}$$

2.17 Vector resolute of $\boldsymbol{a} \perp \boldsymbol{b}$

$$\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u} \text{ where } \boldsymbol{u} \text{ is projection } \boldsymbol{a} \text{ on } \boldsymbol{b}$$

2.18 Vector proofs

2.18.1 Concurrent lines

≥ 3 lines intersect at a single point

2.18.2 Collinear points

≥ 3 points lie on the same line
$\implies \vec{OC} = \lambda\vec{OA} + \mu\vec{OB}$ where $\lambda + \mu = 1$. If C is between $\vec{AB}$, then $0 < \mu < 1$
Points A, B, C are collinear iff $\vec{AC} = m\vec{AB}$ where $m \neq 0$

2.18.3 Useful vector properties

- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then $\boldsymbol{b} = k\boldsymbol{a}$ for some $k \in \mathbb{R} \setminus \{0\}$
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel with at least one point in common, then they lie on the same straight line
- Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if $\boldsymbol{a}\cdot\boldsymbol{b} = 0$
- $\boldsymbol{a}\cdot\boldsymbol{a} = |\boldsymbol{a}|^2$

2.19 Linear dependence

Vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are linearly dependent if they are non-parallel and:

$$k\boldsymbol{a} + l\boldsymbol{b} + m\boldsymbol{c} = 0$$

$$\therefore \boldsymbol{c} = m\boldsymbol{a} + n\boldsymbol{b} \quad \text{(simultaneous)}$$

$\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$ are linearly independent if no vector in the set is expressible as a linear combination of other vectors in set, or if they are parallel.

Vector $\boldsymbol{w}$ is a linear combination of vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$

2.20 Three-dimensional vectors

Right-hand rule for axes: z is up or out of page.

2.21 Parametric vectors

Parametric equation of line through point (x_0, y_0, z_0) and parallel to $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ is:

$$\begin{cases} x = x_o + a \cdot t \\ y = y_0 + b \cdot t \\ z = z_0 + c \cdot t \end{cases} \tag{2}$$