

# 1 Complex numbers

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

Cartesian form:  $a + bi$

Polar form:  $r \operatorname{cis} \theta$

## Operations

	Cartesian	Polar
$z_1 \pm z_2$	$(a \pm c)(b \pm d)i$	convert to $a + bi$
$+k \times z$	$ka \pm kbi$	$kr \operatorname{cis} \theta$
$-k \times z$		$kr \operatorname{cis}(\theta \pm \pi)$
$z_1 \cdot z_2$	$ac - bd + (ad + bc)i$	$r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$
$z_1 \div z_2$	$(z_1 \bar{z}_2) \div  z_2 ^2$	$\left(\frac{r_1}{r_2}\right) \operatorname{cis}(\theta_1 - \theta_2)$

## Scalar multiplication in polar form

For  $k \in \mathbb{R}^+$ :

$$k(r \operatorname{cis} \theta) = kr \operatorname{cis} \theta$$

For  $k \in \mathbb{R}^-$ :

$$k(r \operatorname{cis} \theta) = kr \operatorname{cis} \begin{cases} \theta - \pi & |0 < \operatorname{Arg}(z) \leq \pi \\ \theta + \pi & |-\pi < \operatorname{Arg}(z) \leq 0 \end{cases}$$

## Conjugate

$$\begin{aligned} \bar{z} &= a \mp bi \\ &= r \operatorname{cis}(-\theta) \end{aligned}$$

On CAS: `conjg(a+bi)`

## Properties

$$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$$

$$\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$$

$$\overline{kz} = k\bar{z} \quad | \quad k \in \mathbb{R}$$

$$\begin{aligned} z\bar{z} &= (a + bi)(a - bi) \\ &= a^2 + b^2 \\ &= |z|^2 \end{aligned}$$

## Modulus

$$|z| = |\vec{Oz}| = \sqrt{a^2 + b^2}$$

## Properties

$$|z_1 z_2| = |z_1| |z_2|$$

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$$

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

## Multiplicative inverse

$$\begin{aligned} z^{-1} &= \frac{a - bi}{a^2 + b^2} \\ &= \frac{\bar{z}}{|z|^2} a \\ &= r \operatorname{cis}(-\theta) \end{aligned}$$

## Dividing over $\mathbb{C}$

$$\begin{aligned} \frac{z_1}{z_2} &= z_1 z_2^{-1} \\ &= \frac{z_1 \bar{z}_2}{|z_2|^2} \\ &= \frac{(a + bi)(c - di)}{c^2 + d^2} \end{aligned}$$

(rationalise denominator)

## Polar form

$$\begin{aligned} z &= r \operatorname{cis} \theta \\ &= r(\cos \theta + i \sin \theta) \end{aligned}$$

- $r = |z| = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}$
- $\theta = \operatorname{arg}(z)$     On CAS: `arg(a+bi)`
- $\operatorname{Arg}(z) \in (-\pi, \pi)$     (principal argument)
- Convert on CAS:  
`compToTrig(a+bi)  $\iff$  cExpand{r.cisX}`
- Multiple representations:  
 $r \operatorname{cis} \theta = r \operatorname{cis}(\theta + 2n\pi)$  with  $n \in \mathbb{Z}$  revolutions
- $\operatorname{cis} \pi = -1, \quad \operatorname{cis} 0 = 1$

## de Moivres' theorem

$$(r \operatorname{cis} \theta)^n = r^n \operatorname{cis}(n\theta) \text{ where } n \in \mathbb{Z}$$

### Complex polynomials

Include  $\pm$  for all solutions, incl. imaginary

Sum of squares	$z^2 + a^2 = z^2 - (ai)^2$ $= (z + ai)(z - ai)$
Sum of cubes	$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
Division	$P(z) = D(z)Q(z) + R(z)$
Remainder theorem	Let $\alpha \in \mathbb{C}$ . Remainder of $P(z) \div (z - \alpha)$ is $P(\alpha)$
Factor theorem	$z - \alpha$ is a factor of $P(z) \iff P(\alpha) = 0$ for $\alpha \in \mathbb{C}$
Conjugate root theorem	$P(z) = 0$ at $z = a \pm bi \implies$ both $z_1$ and $\bar{z}_1$ are solutions

### $n$ th roots

$n$ th roots of  $z = r \text{ cis } \theta$  are:

$$z = r^{\frac{1}{n}} \text{ cis } \left( \frac{\theta + 2k\pi}{n} \right)$$

- Same modulus for all solutions
- Arguments separated by  $\frac{2\pi}{n} \therefore$  there are  $n$  roots
- If one square root is  $a + bi$ , the other is  $-a - bi$
- Give one implicit  $n$ th root  $z_1$ , function is  $z = z_1^n$
- Solutions of  $z^n = a$  where  $a \in \mathbb{C}$  lie on the circle  $x^2 + y^2 = \left(|a|^{\frac{1}{n}}\right)^2$  (intervals of  $\frac{2\pi}{n}$ )

For  $0 = az^2 + bz + c$ , use quadratic formula:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

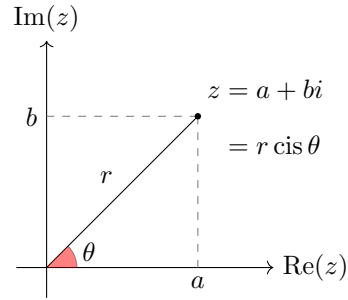
### Fundamental theorem of algebra

A polynomial of degree  $n$  can be factorised into  $n$  linear factors in  $\mathbb{C}$ :

$$\implies P(z) = a_n(z - \alpha_1)(z - \alpha_2)(z - \alpha_3) \dots (z - \alpha_n)$$

where  $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n \in \mathbb{C}$

### Argand planes



- Multiplication by  $i \implies$  CCW rotation of  $\frac{\pi}{2}$
- Addition:  $z_1 + z_2 \equiv \vec{Oz_1} + \vec{Oz_2}$

### Sketching complex graphs

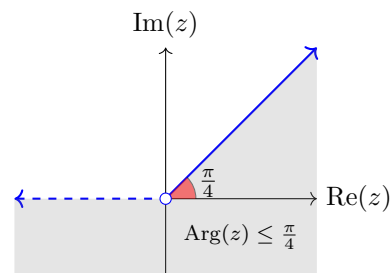
#### Linear

- $\text{Re}(z) = c$  or  $\text{Im}(z) = c$  (perpendicular bisector)
- $\text{Im}(z) = m \text{Re}(z)$
- $|z + a| = |z + b| \implies 2(a - b)x = b^2 - a^2$   
Geometric: equidistant from  $a, b$

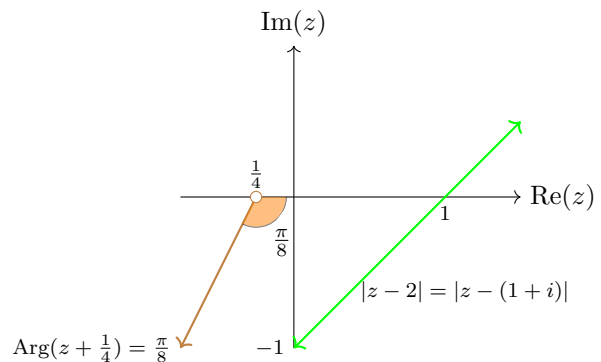
#### Circles

- $|z - z_1|^2 = c^2|z_2 + 2|^2$
- $|z - (a + bi)| = c \implies (x - a)^2 + (y - b)^2 = c^2$

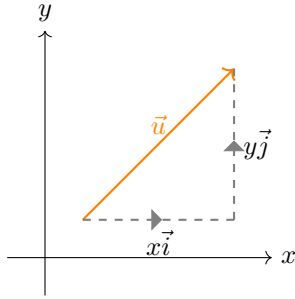
#### Loci $\text{Arg}(z) < \theta$



#### Rays $\text{Arg}(z - b) = \theta$



## 2 Vectors



### Column notation

$$\begin{bmatrix} x \\ y \end{bmatrix} \iff xi + yj$$

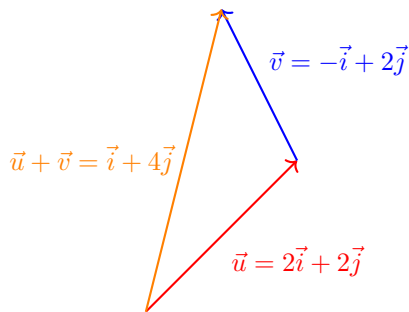
$$\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} \text{ between } A(x_1, y_1), B(x_2, y_2)$$

### Scalar multiplication

$$k \cdot (xi + yj) = kxi + kyj$$

For  $k \in \mathbb{R}^-$ , direction is reversed

### Vector addition



$$(xi + yj) \pm (ai + bj) = (x \pm a)i + (y \pm b)j$$

- Draw each vector head to tail then join lines
- Addition is commutative (parallelogram)
- $u - v = u + (-v) \implies \vec{AB} = b - a$

### Magnitude

$$|(xi + yj)| = \sqrt{x^2 + y^2}$$

### Parallel vectors

$$u \parallel v \iff u = kv \text{ where } k \in \mathbb{R} \setminus \{0\}$$

For parallel vectors  $a$  and  $b$ :

$$a \cdot b = \begin{cases} |a||b| & \text{if same direction} \\ -|a||b| & \text{if opposite directions} \end{cases}$$

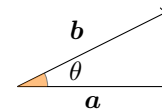
### Perpendicular vectors

$$a \perp b \iff a \cdot b = 0 \quad (\text{since } \cos 90 = 0)$$

### Unit vector $|\hat{a}| = 1$

$$\hat{a} = \frac{1}{|a|}a = a \cdot |a|$$

### Scalar product $a \cdot b$



$$a \cdot b = a_1b_1 + a_2b_2 = |a||b| \cos \theta \quad (0 \leq \theta \leq \pi) - \text{from cosine rule}$$

On CAS: dotP([a b c], [d e f])

### Properties

1.  $k(a \cdot b) = (ka) \cdot b = a \cdot (kb)$
2.  $a \cdot 0 = 0$
3.  $a \cdot (b + c) = a \cdot b + a \cdot c$
4.  $i \cdot i = j \cdot j = k \cdot k = 1$
5.  $a \cdot b = 0 \implies a \perp b$
6.  $a \cdot a = |a|^2 = a^2$

### Angle between vectors

$$\cos \theta = \frac{a \cdot b}{|a||b|} = \frac{a_1b_1 + a_2b_2}{|a||b|}$$

On CAS: angle([a b c], [a b c])

(Action  $\rightarrow$  Vector  $\rightarrow$  Angle)

**Angle between vector and axis**

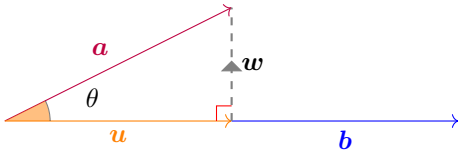
For  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$  which makes angles  $\alpha, \beta, \gamma$  with positive side of  $x, y, z$  axes:

$$\cos \alpha = \frac{a_1}{|\mathbf{a}|}, \quad \cos \beta = \frac{a_2}{|\mathbf{a}|}, \quad \cos \gamma = \frac{a_3}{|\mathbf{a}|}$$

On CAS: `angle([a b c], [1 0 0])`

for angle between  $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$  and  $x$ -axis

**Projections & resolutes**



$\parallel \mathbf{b}$  (vector projection/resolute)

$$\begin{aligned} \mathbf{u} &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2} \mathbf{b} \\ &= \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \right) \left( \frac{\mathbf{b}}{|\mathbf{b}|} \right) \\ &= (\mathbf{a} \cdot \hat{\mathbf{b}}) \hat{\mathbf{b}} \end{aligned}$$

$\perp \mathbf{b}$  (perpendicular projection)

$$\mathbf{w} = \mathbf{a} - \mathbf{u}$$

$|\mathbf{u}|$  (scalar projection/resolute)

$$\begin{aligned} s &= |\mathbf{u}| \\ &= \mathbf{a} \cdot \hat{\mathbf{b}} \\ &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \\ &= |\mathbf{a}| \cos \theta \end{aligned}$$

Rectangular ( $\parallel, \perp$ ) components

$$\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b} + \left( \mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b} \right)$$

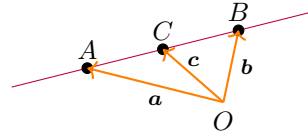
**Vector proofs**

**Concurrent:** intersection of  $\geq 3$  lines



**Collinear points**

$\geq 3$  points lie on the same line



e.g. Prove that

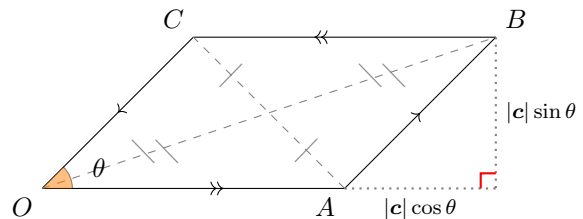
$$\begin{aligned} \vec{AC} = m\vec{AB} &\iff \mathbf{c} = (1 - m)\mathbf{a} + m\mathbf{b} \\ &\implies \mathbf{c} = \vec{OA} + \vec{AC} \\ &= \vec{OA} + m\vec{AB} \\ &= \mathbf{a} + m(\mathbf{b} - \mathbf{a}) \\ &= \mathbf{a} + m\mathbf{b} - m\mathbf{a} \\ &= (1 - m)\mathbf{a} + m\mathbf{b} \end{aligned}$$

Also,  $\implies \vec{OC} = \lambda\vec{OA} + \mu\vec{OB}$

where  $\lambda + \mu = 1$

If  $C$  lies along  $\vec{AB}$ ,  $\implies 0 < \mu < 1$

**Parallelograms**



- Diagonals  $\vec{OB}, \vec{AC}$  bisect each other
- If diagonals are equal length, it is a rectangle
- $|\vec{OB}|^2 + |\vec{CA}|^2 = |\vec{OA}|^2 + |\vec{AB}|^2 + |\vec{CB}|^2 + |\vec{OC}|^2$
- Area =  $\mathbf{c} \cdot \mathbf{a}$

**Useful vector properties**

- $\mathbf{a} \parallel \mathbf{b} \implies \mathbf{b} = k\mathbf{a}$  for some  $k \in \mathbb{R} \setminus \{0\}$
- If  $\mathbf{a}$  and  $\mathbf{b}$  are parallel with at least one point in common, then they lie on the same straight line
- $\mathbf{a} \perp \mathbf{b} \iff \mathbf{a} \cdot \mathbf{b} = 0$
- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

### Linear dependence

$a, b, c$  are linearly dependent if they are  $\nparallel$  and:

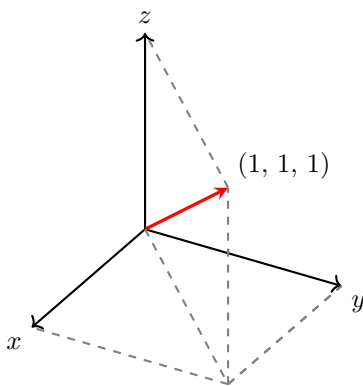
$$0 = ka + lb + mc$$

$$\therefore c = ma + nb \quad (\text{simultaneous})$$

$a, b,$  and  $c$  are linearly independent if no vector in the set is expressible as a linear combination of other vectors in set, or if they are parallel.

### Three-dimensional vectors

Right-hand rule for axes:  $z$  is up or out of page.



### Parametric vectors

Parametric equation of line through point  $(x_0, y_0, z_0)$  and parallel to  $ai + bj + ck$  is:

$$\begin{cases} x = x_0 + a \cdot t \\ y = y_0 + b \cdot t \\ z = z_0 + c \cdot t \end{cases}$$

## 3 Circular functions

$\sin(bx)$  or  $\cos(bx)$ : period =  $\frac{2\pi}{b}$

$\tan(nx)$ : period =  $\frac{\pi}{n}$

asymptotes at  $x = \frac{(2k+1)\pi}{2n} \mid k \in \mathbb{Z}$

### Reciprocal functions

#### Cosecant

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta} \mid \sin \theta \neq 0$$

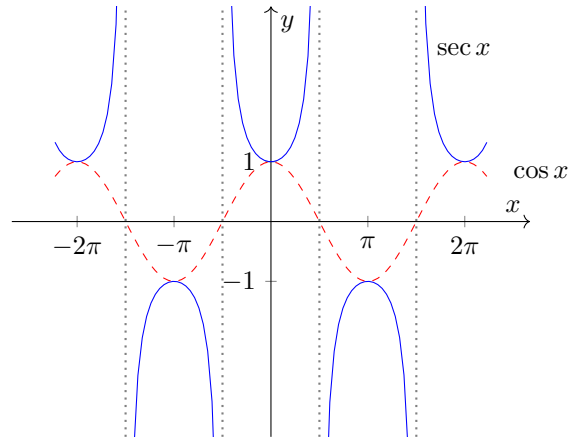
- Domain =  $\mathbb{R} \setminus n\pi : n \in \mathbb{Z}$

- Range =  $\mathbb{R} \setminus (-1, 1)$

- Turning points at  $\theta = \frac{(2n+1)\pi}{2} \mid n \in \mathbb{Z}$

- Asymptotes at  $\theta = n\pi \mid n \in \mathbb{Z}$

#### Secant



$$\sec \theta = \frac{1}{\cos \theta} \mid \cos \theta \neq 0$$

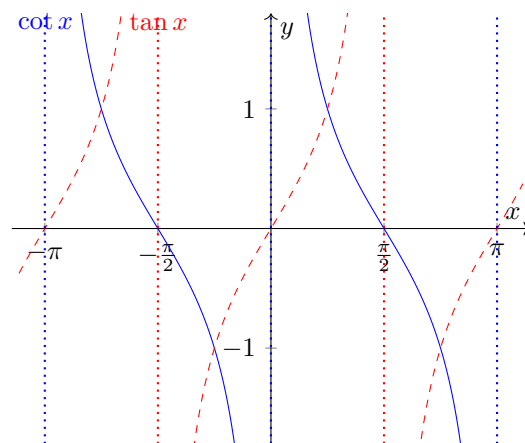
- Domain =  $\mathbb{R} \setminus \frac{(2n+1)\pi}{2} : n \in \mathbb{Z}$

- Range =  $\mathbb{R} \setminus (-1, 1)$

- Turning points at  $\theta = n\pi \mid n \in \mathbb{Z}$

- Asymptotes at  $\theta = \frac{(2n+1)\pi}{2} \mid n \in \mathbb{Z}$

#### Cotangent



$$\cot \theta = \frac{\cos \theta}{\sin \theta} \mid \sin \theta \neq 0$$

- Domain =  $\mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$

- Range =  $\mathbb{R}$

- Asymptotes at  $\theta = n\pi \mid n \in \mathbb{Z}$

## Symmetry properties

$$\sec(\pi \pm x) = -\sec x$$

$$\sec(-x) = \sec x$$

$$\operatorname{cosec}(\pi \pm x) = \mp \operatorname{cosec} x$$

$$\operatorname{cosec}(-x) = -\operatorname{cosec} x$$

$$\cot(\pi \pm x) = \pm \cot x$$

$$\cot(-x) = -\cot x$$

## Complementary properties

$$\sec\left(\frac{\pi}{2} - x\right) = \operatorname{cosec} x$$

$$\operatorname{cosec}\left(\frac{\pi}{2} - x\right) = \sec x$$

$$\cot\left(\frac{\pi}{2} - x\right) = \tan x$$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x$$

## Pythagorean identities

$$1 + \cot^2 x = \operatorname{cosec}^2 x, \quad \text{where } \sin x \neq 0$$

$$1 + \tan^2 x = \sec^2 x, \quad \text{where } \cos x \neq 0$$

## Compound angle formulas

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

## Double angle formulas

$$\cos 2x = \cos^2 x - \sin^2 x$$

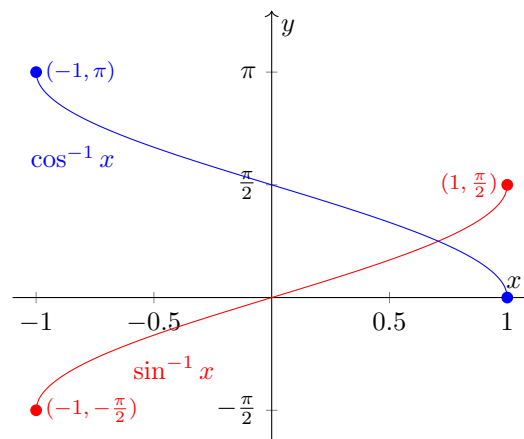
$$= 1 - 2\sin^2 x$$

$$= 2\cos^2 x - 1$$

$$\sin 2x = 2\sin x \cos x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

## Inverse circular functions



Inverse functions:  $f(f^{-1}(x)) = x$  (restrict domain)

$$\sin^{-1} : [-1, 1] \rightarrow \mathbb{R}, \quad \sin^{-1} x = y$$

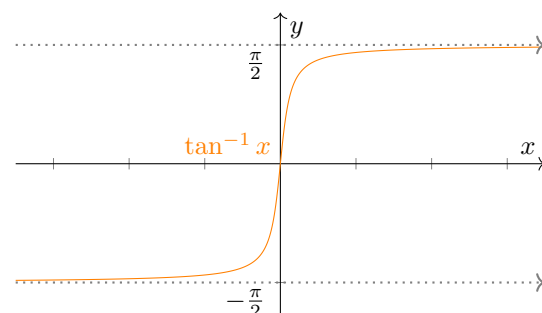
where  $\sin y = x$ ,  $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\cos^{-1} : [-1, 1] \rightarrow \mathbb{R}, \quad \cos^{-1} x = y$$

where  $\cos y = x$ ,  $y \in [0, \pi]$

$$\tan^{-1} : \mathbb{R} \rightarrow \mathbb{R}, \quad \tan^{-1} x = y$$

where  $\tan y = x$ ,  $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



## 4 Differential calculus

### Limits

$$\lim_{x \rightarrow a} f(x)$$

$L^-$ ,  $L^+$  limit from below/above

$\lim_{x \rightarrow a} f(x)$  limit of a point

For solving  $x \rightarrow \infty$ , put all  $x$  terms in denominators

e.g.

$$\lim_{x \rightarrow \infty} \frac{2x + 3}{x - 2} = \frac{2 + \frac{3}{x}}{1 - \frac{2}{x}} = \frac{2}{1} = 2$$

### Limit theorems

1. For constant function  $f(x) = k$ ,  $\lim_{x \rightarrow a} f(x) = k$
2.  $\lim_{x \rightarrow a} (f(x) \pm g(x)) = F \pm G$
3.  $\lim_{x \rightarrow a} (f(x) \times g(x)) = F \times G$
4.  $\therefore \lim_{x \rightarrow a} c \times f(x) = cF$  where  $c = \text{constant}$
5.  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{F}{G}, G \neq 0$
6.  $f(x)$  is continuous  $\iff L^- = L^+ = f(x) \forall x$

### Gradients of secants and tangents

**Secant (chord)** - line joining two points on curve

**Tangent** - line that intersects curve at one point

### First principles derivative

$$f'(x) = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

### Logarithmic identities

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b x^n = n \log_b x$$

$$\log_b y^{x^n} = x^n \log_b y$$

### Index identities

$$b^{m+n} = b^m \cdot b^n$$

$$(b^m)^n = b^{m \cdot n}$$

$$(b \cdot c)^n = b^n \cdot c^n$$

$$a^m \div a^n = a^{m-n}$$

## Derivative rules

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\sin ax$	$a \cos ax$
$\cos x$	$-\sin x$
$\cos ax$	$-a \sin ax$
$\tan f(x)$	$f^2(x) \sec^2 f(x)$
$e^x$	$e^x$
$e^{ax}$	$ae^{ax}$
$ax^{nx}$	$an \cdot e^{nx}$
$\log_e x$	$\frac{1}{x}$
$\log_e ax$	$\frac{1}{x}$
$\log_e f(x)$	$\frac{f'(x)}{f(x)}$
$\sin(f(x))$	$f'(x) \cdot \cos(f(x))$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\text{sqr}t{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\frac{d}{dy} f(y)$	$\frac{1}{\frac{dx}{dy}}$ (reciprocal)
$uv$	$u \frac{dv}{dx} + v \frac{du}{dx}$ (product rule)
$\frac{u}{v}$	$\frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$ (quotient rule)
$f(g(x))$	$f'(g(x)) \cdot g'(x)$

### Reciprocal derivatives

$$\frac{1}{\frac{dy}{dx}} = \frac{dx}{dy}$$

### Differentiating $x = f(y)$

$$\begin{aligned} &\text{Find } \frac{dx}{dy} \\ \text{Then, } &\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \\ \implies &\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \\ \therefore &\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \end{aligned}$$

### Second derivative

$$\begin{aligned} f(x) &\longrightarrow f'(x) \longrightarrow f''(x) \\ \implies y &\longrightarrow \frac{dy}{dx} \longrightarrow \frac{d^2y}{dx^2} \end{aligned}$$

Order of polynomial  $n$ th derivative decrements each time the derivative is taken

### Points of Inflection

Stationary point - i.e.  $f'(x) = 0$

Point of inflection - max |gradient| (i.e.  $f'' = 0$ )

- if  $f'(a) = 0$  and  $f''(a) > 0$ , then point  $(a, f(a))$  is a local min (curve is concave up)
- if  $f'(a) = 0$  and  $f''(a) < 0$ , then point  $(a, f(a))$  is local max (curve is concave down)
- if  $f''(a) = 0$ , then point  $(a, f(a))$  is a point of inflection
- if also  $f'(a) = 0$ , then it is a stationary point of inflection

### Implicit Differentiation

Used for differentiating circles etc.

If  $p$  and  $q$  are expressions in  $x$  and  $y$  such that  $p = q$ , for all  $x$  and  $y$ , then:

$$\frac{dp}{dx} = \frac{dq}{dx} \quad \text{and} \quad \frac{dp}{dy} = \frac{dq}{dy}$$

#### On CAS:

Action  $\rightarrow$  Calculation  $\rightarrow$  `impDiff(y^2+ax=5, x, y)`

Returns  $y' = \dots$

### Integration

$$\int f(x) \cdot dx = F(x) + c \quad \text{where } F'(x) = f(x)$$

### Integral laws

$f(x)$	$\int f(x) \cdot dx$
$k$ (constant)	$kx + c$
$x^n$	$\frac{1}{n+1}x^{n+1}$
$ax^{-n}$	$a \cdot \log_e  x  + c$
$\frac{1}{ax+b}$	$\frac{1}{a} \log_e(ax+b) + c$
$(ax+b)^n$	$\frac{1}{a(n+1)}(ax+b)^{n+1} + c \mid n \neq -1$
$(ax+b)^{-1}$	$\frac{1}{a} \log_e  ax+b  + c$
$e^{kx}$	$\frac{1}{k}e^{kx} + c$
$e^k$	$e^k x + c$
$\sin kx$	$-\frac{1}{k} \cos(kx) + c$
$\cos kx$	$\frac{1}{k} \sin(kx) + c$
$\sec^2 kx$	$\frac{1}{k} \tan(kx) + c$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1} \frac{x}{a} + c \mid a > 0$
$\frac{-1}{\sqrt{a^2-x^2}}$	$\cos^{-1} \frac{x}{a} + c \mid a > 0$
$\frac{a}{a^2-x^2}$	$\tan^{-1} \frac{x}{a} + c$
$\frac{f'(x)}{f(x)}$	$\log_e f(x) + c$
$\int f(u) \cdot \frac{du}{dx} \cdot dx$	$\int f(u) \cdot du$ (substitution)
$f(x) \cdot g(x)$	$\int [f'(x) \cdot g(x)]dx + \int [g'(x)f(x)]dx$

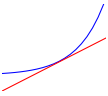
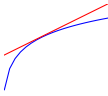
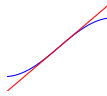
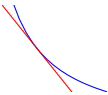
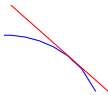
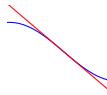
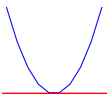
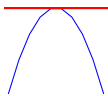
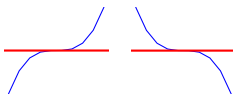
Note  $\sin^{-1} \frac{x}{a} + \cos^{-1} \frac{x}{a}$  is constant  $\forall x \in (-a, a)$

### Definite integrals

$$\int_a^b f(x) \cdot dx = [F(x)]_a^b = F(b) - F(a)$$

- Signed area enclosed by  $y = f(x)$ ,  $y = 0$ ,  $x = a$ ,  $x = b$ .
- *Integrand* is  $f$ .



	$\frac{d^2y}{dx^2} > 0$	$\frac{d^2y}{dx^2} < 0$	$\frac{d^2y}{dx^2} = 0$ (inflection)
$\frac{dy}{dx} > 0$			
	Rising (concave up)	Rising (concave down)	Rising inflection point
$\frac{dy}{dx} < 0$			
	Falling (concave up)	Falling (concave down)	Falling inflection point
$\frac{dy}{dx} = 0$			
	Local minimum	Local maximum	Stationary inflection point

**Properties**

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$\int_a^a f(x) dx = 0$$

$$\int_a^b k \cdot f(x) dx = k \int_a^b f(x) dx$$

$$\int_a^b f(x) \pm g(x) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

**Integration by substitution**

$$\int f(u) \frac{du}{dx} \cdot dx = \int f(u) \cdot du$$

Note  $f(u)$  must be 1:1  $\implies$  one  $x$  for each  $y$

e.g. for  $y = \int (2x+1)\sqrt{x+4} \cdot dx$

let  $u = x + 4$

$$\implies \frac{du}{dx} = 1$$

$$\implies x = u - 4$$

then  $y = \int (2(u-4)+1)u^{\frac{1}{2}} \cdot du$

(solve as normal integral)

**Definite integrals by substitution**

For  $\int_a^b f(x) \frac{du}{dx} \cdot dx$ , evaluate new  $a$  and  $b$  for  $f(u) \cdot du$ .

**Trigonometric integration**

$$\sin^m x \cos^n x \cdot dx$$

**$m$  is odd:**  $m = 2k + 1$  where  $k \in \mathbb{Z}$

$$\implies \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x$$

Substitute  $u = \cos x$

**$n$  is odd:**  $n = 2k + 1$  where  $k \in \mathbb{Z}$

$$\implies \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x$$

Substitute  $u = \sin x$

**$m$  and  $n$  are even:** use identities...

- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

- $\sin 2x = 2 \sin x \cos x$

**Partial fractions**

**On CAS:**

Action  $\rightarrow$  Transformation  $\rightarrow$  expand/combine

Interactive  $\rightarrow$  Transformation  $\rightarrow$  Expand  $\rightarrow$  Partial

## Graphing integrals on CAS

In main: Interactive → Calculation →  $\int$  (→ Definite)

Restrictions: Define  $f(x)=..$  then  $f(x)|x>..$

## Applications of antidifferentiation

- $x$ -intercepts of  $y = f(x)$  identify  $x$ -coordinates of stationary points on  $y = F(x)$
- nature of stationary points is determined by sign of  $y = f(x)$  on either side of its  $x$ -intercepts
- if  $f(x)$  is a polynomial of degree  $n$ , then  $F(x)$  has degree  $n + 1$

To find stationary points of a function, substitute  $x$  value of given point into derivative. Solve for  $\frac{dy}{dx} = 0$ . Integrate to find original function.

## Solids of revolution

Approximate as sum of infinitesimally-thick cylinders

### Rotation about $x$ -axis

$$\begin{aligned} V &= \int_{x=a}^{x=b} \pi y^2 dx \\ &= \pi \int_a^b (f(x))^2 dx \end{aligned}$$

### Rotation about $y$ -axis

$$\begin{aligned} V &= \int_{y=a}^{y=b} \pi x^2 dy \\ &= \pi \int_a^b (f(y))^2 dy \end{aligned}$$

### Regions not bound by $y = 0$

$$V = \pi \int_a^b f(x)^2 - g(x)^2 dx$$

where  $f(x) > g(x)$

## Length of a curve

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad (\text{Cartesian})$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (\text{parametric})$$

## On CAS:

Evaluate formula,

or Interactive → Calculation → Line → arcLen

## Rates

### Gradient at a point on parametric curve

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} \mid \frac{dx}{dt} \neq 0 \quad (\text{chain rule})$$

$$\frac{d^2}{dx^2} = \frac{d(y')}{dx} = \frac{dy'}{dt} \div \frac{dx}{dt} \mid y' = \frac{dy}{dx}$$

## Rational functions

$$f(x) = \frac{P(x)}{Q(x)} \quad \text{where } P, Q \text{ are polynomial functions}$$

### Addition of ordinates

- when two graphs have the same ordinate,  $y$ -coordinate is double the ordinate
- when two graphs have opposite ordinates,  $y$ -coordinate is 0 i.e. ( $x$ -intercept)
- when one of the ordinates is 0, the resulting ordinate is equal to the other ordinate

## Fundamental theorem of calculus

If  $f$  is continuous on  $[a, b]$ , then

$$\int_a^b f(x) dx = F(b) - F(a)$$

where  $F = \int f dx$

## Differential equations

**Order** - highest power inside derivative

**Degree** - highest power of highest derivative

e.g.  $\left(\frac{dy^2}{dx^2}x\right)^3$  order 2, degree 3

### Verifying solutions

Start with  $y = \dots$ , and differentiate. Substitute into original equation.

**Function of the dependent variable**

If  $\frac{dy}{dx} = g(y)$ , then  $\frac{dx}{dy} = 1 \div \frac{dy}{dx} = \frac{1}{g(y)}$ . Integrate both sides to solve equation. Only add  $c$  on one side. Express  $e^c$  as  $A$ .

**Mixing problems**

$$\left(\frac{dm}{dt}\right)_{\Sigma} = \left(\frac{dm}{dt}\right)_{\text{in}} - \left(\frac{dm}{dt}\right)_{\text{out}}$$

**Separation of variables**

If  $\frac{dy}{dx} = f(x)g(y)$ , then:

$$\int f(x) dx = \int \frac{1}{g(y)} dy$$

**Euler's method for solving DEs**

$$\frac{f(x+h) - f(x)}{h} \approx f'(x) \quad \text{for small } h$$

$$\implies f(x+h) \approx f(x) + hf'(x)$$

**5 Kinematics & Mechanics****Constant acceleration**

- **Position** - relative to origin
- **Displacement** - relative to starting point

**Velocity-time graphs**

- Displacement: *signed* area between graph and  $t$  axis
- Distance travelled: *total* area between graph and  $t$  axis

$$\text{acceleration} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v \frac{dv}{dx} = \frac{d}{dx} \left( \frac{1}{2}v^2 \right)$$

	no
$v = u + at$	$x$
$v^2 = u^2 + 2as$	$t$
$s = \frac{1}{2}(v+u)t$	$a$
$s = ut + \frac{1}{2}at^2$	$v$
$s = vt - \frac{1}{2}at^2$	$u$

$$v_{\text{avg}} = \frac{\Delta \text{position}}{\Delta t}$$

$$\begin{aligned} \text{speed} &= |\text{velocity}| \\ &= \sqrt{v_x^2 + v_y^2 + v_z^2} \end{aligned}$$

**Distance travelled between  $t = a \rightarrow t = b$ :**

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$$

**Shortest distance between  $\mathbf{r}(t_0)$  and  $\mathbf{r}(t_1)$ :**

$$= |\mathbf{r}(t_1) - \mathbf{r}(t_2)|$$

**Vector functions**

$$\mathbf{r}(t) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

- If  $\mathbf{r}(t) \equiv$  position with time, then the graph of endpoints of  $\mathbf{r}(t) \equiv$  Cartesian path
- Domain of  $\mathbf{r}(t)$  is the range of  $x(t)$
- Range of  $\mathbf{r}(t)$  is the range of  $y(t)$

**Vector calculus****Derivative**

Let  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ . If both  $x(t)$  and  $y(t)$  are differentiable, then:

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$