1 Complex numbers

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

$$\text{Cartesian form: } a + bi$$
$$\text{Polar form: } r \operatorname{cis} \theta$$

Operations

	Cartesian	**Polar**
$z_1 \pm z_2$	$(a \pm c)(b \pm d)i$	convert to $a + bi$
$+k \times z$	$ka \pm kbi$	$kr \operatorname{cis} \theta$
$-k \times z$		$kr \operatorname{cis}(\theta \pm \pi)$
$z_1 \cdot z_2$	$ac - bd + (ad + bc)i$	$r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$
$z_1 \div z_2$	$(z_1 \overline{z_2}) \div \lvert z_2 \rvert^2$	$\left(\frac{r_1}{r_2}\right) \operatorname{cis}(\theta_1 - \theta_2)$

Scalar multiplication in polar form

For $k \in \mathbb{R}^+$:

$$k\,(r \operatorname{cis} \theta) = kr \operatorname{cis} \theta$$

For $k \in \mathbb{R}^-$:

$$k\,(r \operatorname{cis} \theta) = kr \operatorname{cis} \left(\begin{cases} \theta - \pi & |0 < \operatorname{Arg}(z) \le \pi \\ \theta + \pi & | - \pi < \operatorname{Arg}(z) \le 0 \end{cases} \right)$$

Conjugate

$$\begin{aligned} \overline{z} &= a \mp bi \\ &= r \operatorname{cis}(-\theta) \end{aligned}$$

On CAS: `conjg(a+bi)`

Properties

$$\begin{aligned} \overline{z_1 \pm z_2} &= \overline{z_1} \pm \overline{z_2} \\ \overline{z_1 \cdot z_2} &= \overline{z_1} \cdot \overline{z_2} \\ \overline{kz} &= k\overline{z} \quad | \quad k \in \mathbb{R} \\ z\overline{z} &= (a + bi)(a - bi) \\ &= a^2 + b^2 \\ &= |z|^2 \end{aligned}$$

Modulus

$$|z| = |\vec{Oz}| = \sqrt{a^2 + b^2}$$

Properties

$$\begin{aligned} |z_1 z_2| &= |z_1||z_2| \\ \left| \frac{z_1}{z_2} \right| &= \frac{|z_1|}{|z_2|} \\ |z_1 + z_2| &\le |z_1| + |z_2| \end{aligned}$$

Multiplicative inverse

$$\begin{aligned} z^{-1} &= \frac{a - bi}{a^2 + b^2} \\ &= \frac{\overline{z}}{|z|^2} a \\ &= r \operatorname{cis}(-\theta) \end{aligned}$$

Dividing over $\mathbb{C}$

$$\begin{aligned} \frac{z_1}{z_2} &= z_1 z_2^{-1} \\ &= \frac{z_1 \overline{z_2}}{|z_2|^2} \\ &= \frac{(a + bi)(c - di)}{c^2 + d^2} \end{aligned}$$

(rationalise denominator)

Polar form

$$\begin{aligned} z &= r \operatorname{cis} \theta \\ &= r(\cos \theta + i \sin \theta) \end{aligned}$$

- $r = |z| = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}$
- $\theta = \arg(z)$ On CAS: `arg(a+bi)`
- $\operatorname{Arg}(z) \in (-\pi, \pi)$ **(principal argument)**
- Convert on CAS:
  `compToTrig(a+bi)` $\iff$ `cExpand{r·cisX}`
- Multiple representations:
  $r \operatorname{cis} \theta = r \operatorname{cis}(\theta + 2n\pi)$ with $n \in \mathbb{Z}$ revolutions
- $\operatorname{cis} \pi = -1, \qquad \operatorname{cis} 0 = 1$

de Moivres' theorem

$$(r \operatorname{cis} \theta)^n = r^n \operatorname{cis}(n\theta) \text{ where } n \in \mathbb{Z}$$

Complex polynomials

Include $\pm$ for all solutions, incl. imaginary	
Sum of squares	$z^2 + a^2 = z^2 - (ai)^2$ $= (z + ai)(z - ai)$
Sum of cubes	$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
Division	$P(z) = D(z)Q(z) + R(z)$
Remainder theorem	Let $\alpha \in \mathbb{C}$. Remainder of $P(z) \div (z - \alpha)$ is $P(\alpha)$
Factor theorem	$z - \alpha$ is a factor of $P(z) \iff P(\alpha) = 0$ for $\alpha \in \mathbb{C}$
Conjugate root theorem	$P(z) = 0$ at $z = a \pm bi$ ($\implies$ both z_1 and $\overline{z_1}$ are solutions)

nth roots

nth roots of $z = r \operatorname{cis} \theta$ are:

$$z = r^{\frac{1}{n}} \operatorname{cis}\left(\frac{\theta + 2k\pi}{n}\right)$$

- Same modulus for all solutions
- Arguments separated by $\frac{2\pi}{n}$ $\therefore$ there are n roots
- If one square root is $a + bi$, the other is $-a - bi$
- Give one implicit nth root z_1, function is $z = z_1^n$
- Solutions of $z^n = a$ where $a \in \mathbb{C}$ lie on the circle $x^2 + y^2 = \left(|a|^{\frac{1}{n}}\right)^2$ (intervals of $\frac{2\pi}{n}$)

For $0 = az^2 + bz + c$, use quadratic formula:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

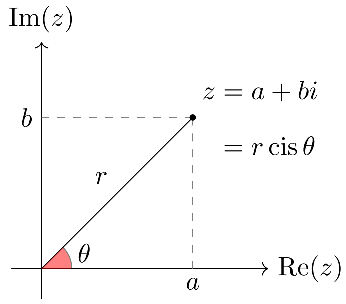
Fundamental theorem of algebra

A polynomial of degree n can be factorised into n linear factors in $\mathbb{C}$:

$$\implies P(z) = a_n(z - \alpha_1)(z - \alpha_2)(z - \alpha_3) \ldots (z - \alpha_n)$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \in \mathbb{C}$

Argand planes



- Multiplication by $i \implies$ CCW rotation of $\frac{\pi}{2}$
- Addition: $z_1 + z_2 \equiv \overrightarrow{Oz_1} + \overrightarrow{Oz_2}$

Sketching complex graphs

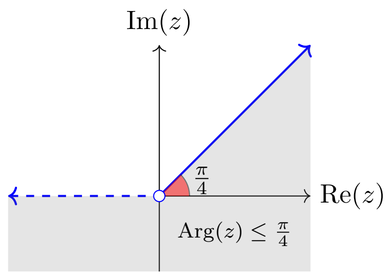
Linear

- $\operatorname{Re}(z) = c$ or $\operatorname{Im}(z) = c$ (perpendicular bisector)
- $\operatorname{Im}(z) = m \operatorname{Re}(z)$
- $|z + a| = |z + b| \implies 2(a - b)x = b^2 - a^2$
  Geometric: equidistant from a, b

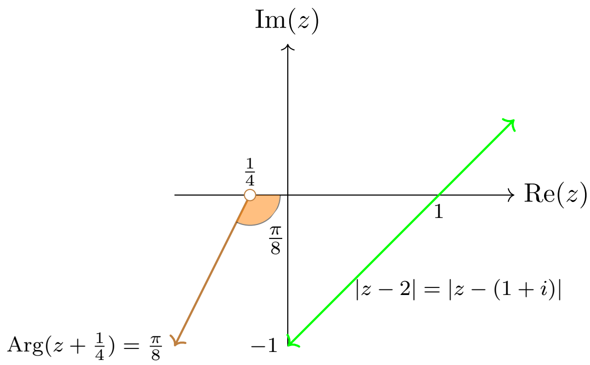
Circles

- $|z - z_1|^2 = c^2 |z_2 + 2|^2$
- $|z - (a + bi)| = c \implies (x - a)^2 +_{(} y - b)^2 = c^2$

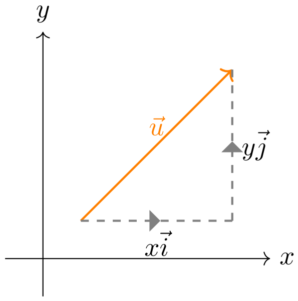
Loci $\operatorname{Arg}(z) < \theta$



Rays $\operatorname{Arg}(z - b) = \theta$



2 Vectors



Column notation

$$\begin{bmatrix} x \\ y \end{bmatrix} \iff x\boldsymbol{i} + y\boldsymbol{j}$$

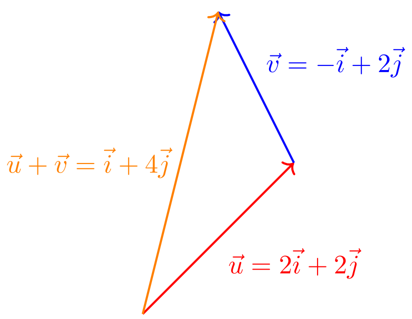
$\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$ between $A(x_1, y_1),\ B(x_2, y_2)$

Scalar multiplication

$$k \cdot (x\boldsymbol{i} + y\boldsymbol{j}) = kx\boldsymbol{i} + ky\boldsymbol{j}$$

For $k \in \mathbb{R}^-$, direction is reversed

Vector addition



$$(x\boldsymbol{i} + y\boldsymbol{j}) \pm (a\boldsymbol{i} + b\boldsymbol{j}) = (x \pm a)\boldsymbol{i} + (y \pm b)\boldsymbol{j}$$

- Draw each vector head to tail then join lines
- Addition is commutative (parallelogram)
- $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v}) \implies \overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

Magnitude

$$|(x\boldsymbol{i} + y\boldsymbol{j})| = \sqrt{x^2 + y^2}$$

Parallel vectors

$$\boldsymbol{u} || \boldsymbol{v} \iff \boldsymbol{u} = k\boldsymbol{v} \text{ where } k \in \mathbb{R} \setminus \{0\}$$

For parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{cases} |\boldsymbol{a}||\boldsymbol{b}| & \text{if same direction} \\ -|\boldsymbol{a}||\boldsymbol{b}| & \text{if opposite directions} \end{cases}$$

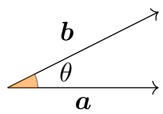
Perpendicular vectors

$$\boldsymbol{a} \perp \boldsymbol{b} \iff \boldsymbol{a} \cdot \boldsymbol{b} = 0 \quad (\text{since } \cos 90 = 0)$$

Unit vector $|\hat{\boldsymbol{a}}| = 1$

$$\begin{aligned} \hat{\boldsymbol{a}} &= \frac{1}{|\boldsymbol{a}|}\boldsymbol{a} \\ &= \boldsymbol{a} \cdot |\boldsymbol{a}| \end{aligned}$$

Scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$



$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= a_1 b_1 + a_2 b_2 \\ &= |\boldsymbol{a}||\boldsymbol{b}| \cos\theta \end{aligned}$$

$(0 \le \theta \le \pi)$ - from cosine rule

On CAS: `dotP([a b c], [d e f])`

Properties

1. $k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (k\boldsymbol{b})$
2. $\boldsymbol{a} \cdot \boldsymbol{0} = 0$
3. $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
4. $\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1$
5. $\boldsymbol{a} \cdot \boldsymbol{b} = 0 \implies \boldsymbol{a} \perp \boldsymbol{b}$
6. $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2 = a^2$

Angle between vectors

$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{a_1 b_1 + a_2 b_2}{|\boldsymbol{a}||\boldsymbol{b}|}$$

On CAS: `angle([a b c], [a b c])`

(Action → Vector →Angle)

Angle between vector and axis

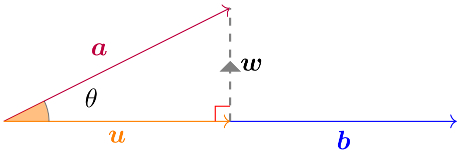
For $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ which makes angles α, β, γ with positive side of x, y, z axes:

$$\cos\alpha = \frac{a_1}{|\boldsymbol{a}|}, \quad \cos\beta = \frac{a_2}{|\boldsymbol{a}|}, \quad \cos\gamma = \frac{a_3}{|\boldsymbol{a}|}$$

On CAS: `angle([a b c], [1 0 0])`

for angle between $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ and x-axis

Projections & resolutes



$\parallel \boldsymbol{b}$ (vector projection/resolute)

$$\begin{aligned}\boldsymbol{u} &= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|^2}\boldsymbol{b} \\ &= \left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|}\right)\left(\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right) \\ &= (\boldsymbol{a}\cdot\hat{\boldsymbol{b}})\hat{\boldsymbol{b}}\end{aligned}$$

$\perp \boldsymbol{b}$ (perpendicular projection)

$$\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u}$$

$|\boldsymbol{u}|$ (scalar projection/resolute)

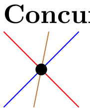
$$\begin{aligned}s &= |\boldsymbol{u}| \\ &= \boldsymbol{a}\cdot\hat{\boldsymbol{b}} \\ &= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|} \\ &= |\boldsymbol{a}|\cos\theta\end{aligned}$$

Rectangular ($\parallel, \perp$) components

$$\boldsymbol{a} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}\cdot\boldsymbol{b}}\boldsymbol{b} + \left(\boldsymbol{a} - \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}\cdot\boldsymbol{b}}\boldsymbol{b}\right)$$

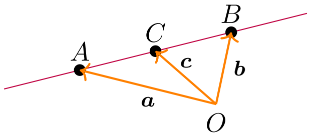
Vector proofs

Concurrent: intersection of ≥ 3 lines



Collinear points

≥ 3 points lie on the same line



e.g. Prove that

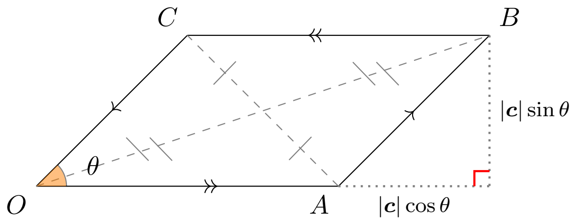
$$\begin{aligned}\overrightarrow{AC} = m\overrightarrow{AB} &\iff \boldsymbol{c} = (1-m)\boldsymbol{a} + m\boldsymbol{b} \\ &\implies \boldsymbol{c} = \overrightarrow{OA} + \overrightarrow{AC} \\ &\qquad = \overrightarrow{OA} + m\overrightarrow{AB} \\ &\qquad = \boldsymbol{a} + m(\boldsymbol{b} - \boldsymbol{a}) \\ &\qquad = \boldsymbol{a} + m\boldsymbol{b} - m\boldsymbol{a} \\ &\qquad = (1-m)\boldsymbol{a} + m\boldsymbol{b}\end{aligned}$$

Also, $\implies \overrightarrow{OC} = \lambda\vec{OA} + \mu\overrightarrow{OB}$

where $\lambda + \mu = 1$

If C lies along $\overrightarrow{AB}$, $\implies 0 < \mu < 1$

Parallelograms



- Diagonals $\overrightarrow{OB}, \overrightarrow{AC}$ bisect each other
- If diagonals are equal length, it is a rectangle
- $|\overrightarrow{OB}|^2 + |\overrightarrow{CA}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{AB}|^2 + |\overrightarrow{CB}|^2 + |\overrightarrow{OC}|^2$
- Area $= \boldsymbol{c}\cdot\boldsymbol{a}$

Useful vector properties

- $\boldsymbol{a} \parallel \boldsymbol{b} \implies \boldsymbol{b} = k\boldsymbol{a}$ for some $k \in \mathbb{R}\setminus\{0\}$
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel with at least one point in common, then they lie on the same straight line
- $\boldsymbol{a} \perp \boldsymbol{b} \iff \boldsymbol{a}\cdot\boldsymbol{b} = 0$
- $\boldsymbol{a}\cdot\boldsymbol{a} = |\boldsymbol{a}|^2$

Linear dependence

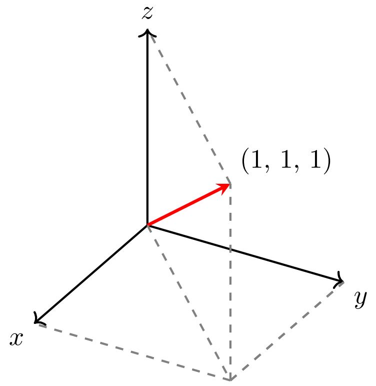
$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are linearly dependent if they are $\nparallel$ and:

$$0 = k\boldsymbol{a} + l\boldsymbol{b} + m\boldsymbol{c}$$
$$\therefore \boldsymbol{c} = m\boldsymbol{a} + n\boldsymbol{b} \quad \text{(simultaneous)}$$

$\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$ are linearly independent if no vector in the set is expressible as a linear combination of other vectors in set, or if they are parallel.

Three-dimensional vectors

Right-hand rule for axes: z is up or out of page.



Parametric vectors

Parametric equation of line through point (x_0, y_0, z_0) and parallel to $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ is:

$$\begin{cases} x = x_o + a \cdot t \\ y = y_0 + b \cdot t \\ z = z_0 + c \cdot t \end{cases}$$

3 Circular functions

$\sin(bx)$ or $\cos(bx)$: period $= \frac{2\pi}{b}$
$\tan(nx)$: period $= \frac{\pi}{n}$
asymptotes at $x = \frac{(2k+1)\pi}{2n} \mid k \in \mathbb{Z}$

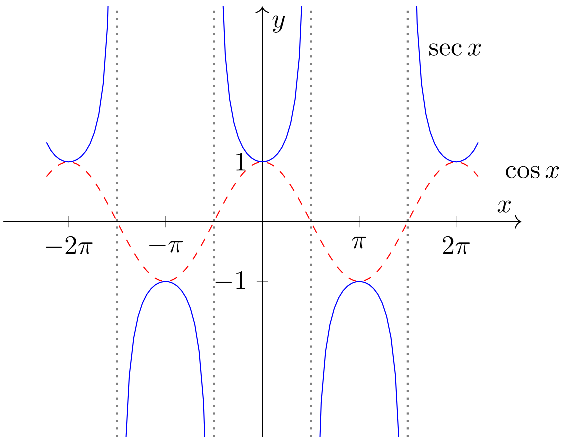
Reciprocal functions

Cosecant

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \mid \sin\theta \neq 0$$

- **Domain** $= \mathbb{R} \setminus n\pi : n \in \mathbb{Z}$
- **Range** $= \mathbb{R} \setminus (-1, 1)$
- **Turning points** at $\theta = \frac{(2n+1)\pi}{2} \mid n \in \mathbb{Z}$
- **Asymptotes** at $\theta = n\pi \mid n \in \mathbb{Z}$

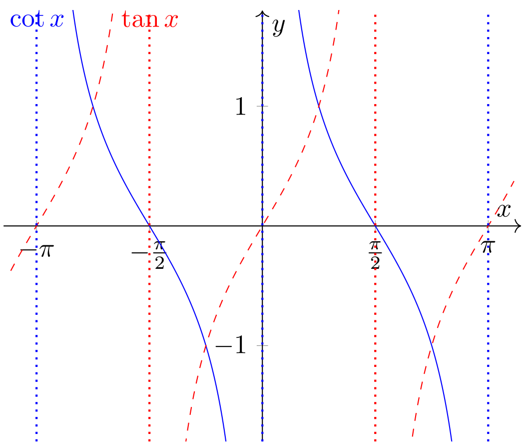
Secant



$$\sec\theta = \frac{1}{\cos\theta} \mid \cos\theta \neq 0$$

- **Domain** $= \mathbb{R} \setminus \frac{(2n+1)\pi}{2} : n \in \mathbb{Z}\}$
- **Range** $= \mathbb{R} \setminus (-1, 1)$
- **Turning points** at $\theta = n\pi \mid n \in \mathbb{Z}$
- **Asymptotes** at $\theta = \frac{(2n+1)\pi}{2} \mid n \in \mathbb{Z}$

Cotangent



$$\cot\theta = \frac{\cos\theta}{\sin\theta} \mid \sin\theta \neq 0$$

- **Domain** $= \mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$
- **Range** $= \mathbb{R}$
- **Asymptotes** at $\theta = n\pi \mid n \in \mathbb{Z}$

Symmetry properties

$$\sec(\pi \pm x) = -\sec x$$
$$\sec(-x) = \sec x$$
$$\operatorname{cosec}(\pi \pm x) = \mp \operatorname{cosec} x$$
$$\operatorname{cosec}(-x) = -\operatorname{cosec} x$$
$$\cot(\pi \pm x) = \pm \cot x$$
$$\cot(-x) = -\cot x$$

Complementary properties

$$\sec\left(\frac{\pi}{2} - x\right) = \operatorname{cosec} x$$
$$\operatorname{cosec}\left(\frac{\pi}{2} - x\right) = \sec x$$
$$\cot\left(\frac{\pi}{2} - x\right) = \tan x$$
$$\tan\left(\frac{\pi}{2} - x\right) = \cot x$$

Pythagorean identities

$$1 + \cot^2 x = \operatorname{cosec}^2 x, \quad \text{where } \sin x \neq 0$$
$$1 + \tan^2 x = \sec^2 x, \quad \text{where } \cos x \neq 0$$

Compound angle formulas

$$\cos(x \pm y) = \cos x + \cos y \mp \sin x \sin y$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

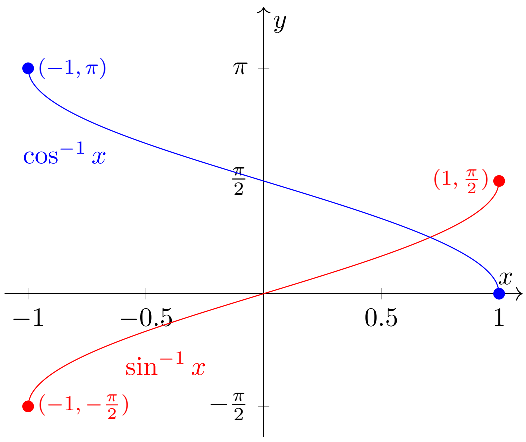
Double angle formulas

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 1 - 2\sin^2 x$$
$$= 2\cos^2 x - 1$$

$$\sin 2x = 2 \sin x \cos x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Inverse circular functions



Inverse functions: $f(f^{-1}(x)) = x$ (restrict domain)

$$\sin^{-1} : [-1, 1] \to \mathbb{R}, \quad \sin^{-1} x = y$$

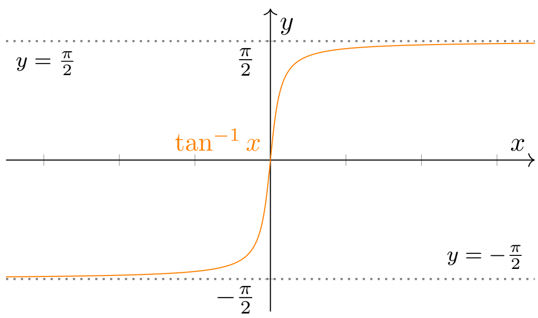
where $\sin y = x$, $y \in [\frac{-\pi}{2}, \frac{\pi}{2}]$

$$\cos^{-1} : [-1, 1] \to \mathbb{R}, \quad \cos^{-1} x = y$$

where $\cos y = x$, $y \in [0, \pi]$

$$\tan^{-1} : \mathbb{R} \to \mathbb{R}, \quad \tan^{-1} x = y$$

where $\tan y = x$, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



4 Differential calculus

Limits

$$\lim_{x \to a} f(x)$$

L^-, L^+	limit from below/above
$\lim_{x \to a} f(x)$	limit of a point

For solving $x \to \infty$, put all x terms in denominators

e.g.

$$\lim_{x \to \infty} \frac{2x+3}{x-2} = \frac{2+\frac{3}{x}}{1-\frac{2}{x}} = \frac{2}{1} = 2$$

Limit theorems

1. For constant function $f(x) = k$, $\lim_{x \to a} f(x) = k$
2. $\lim_{x \to a}(f(x) \pm g(x)) = F \pm G$
3. $\lim_{x \to a}(f(x) \times g(x)) = F \times G$
4. $\therefore \lim_{x \to a} c \times f(x) = cF$ where $c =$ constant
5. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G}, G \neq 0$
6. $f(x)$ is continuous $\iff L^- = L^+ = f(x) \, \forall x$

Gradients of secants and tangents

Secant (chord) - line joining two points on curve

Tangent - line that intersects curve at one point

First principles derivative

$$f'(x) = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

Logarithmic identities

$\log_b(xy) = \log_b x + \log_b y$

$\log_b x^n = n \log_b x$

$\log_b y^{x^n} = x^n \log_b y$

Index identities

$b^{m+n} = b^m \cdot b^n$

$(b^m)^n = b^{m \cdot n}$

$(b \cdot c)^n = b^n \cdot c^n$

$a^m \div a^n = a^{m-n}$

Derivative rules

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\sin ax$	$a \cos ax$
$\cos x$	$-\sin x$
$\cos ax$	$-a \sin ax$
$\tan f(x)$	$f^2(x) \sec^2 f(x)$
e^x	e^x
e^{ax}	ae^{ax}
ax^{nx}	$an \cdot e^{nx}$
$\log_e x$	$\frac{1}{x}$
$\log_e ax$	$\frac{1}{x}$
$\log_e f(x)$	$\frac{f'(x)}{f(x)}$
$\sin(f(x))$	$f'(x) \cdot \cos(f(x))$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{sqrt1-x^2}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\frac{d}{dy} f(y)$	$\frac{1}{\frac{dx}{dy}}$ (reciprocal)
uv	$u\frac{dv}{dx} + v\frac{du}{dx}(productrule)$
$\frac{u}{v}$	$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ (quotient rule)
$f(g(x))$	$f'(g(x)) \cdot g'(x)$

Reciprocal derivatives

$$\frac{1}{\frac{dy}{dx}} = \frac{dx}{dy}$$

Differentiating $x = f(y)$

$$\text{Find } \frac{dx}{dy}$$
$$\text{Then, } \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$
$$\implies \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$
$$\therefore \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Second derivative

$$f(x) \longrightarrow f'(x) \longrightarrow f''(x)$$
$$\implies y \longrightarrow \frac{dy}{dx} \longrightarrow \frac{d^2y}{dx^2}$$

Order of polynomial nth derivative decrements each time the derivative is taken

Points of Inflection

Stationary point - i.e. $f'(x) = 0$
Point of inflection - max |gradient| (i.e. $f'' = 0$)

- if $f'(a) = 0$ and $f''(a) > 0$, then point $(a, f(a))$ is a local min (curve is concave up)
- if $f'(a) = 0$ and $f''(a) < 0$, then point $(a, f(a))$ is local max (curve is concave down)
- if $f''(a) = 0$, then point $(a, f(a))$ is a point of inflection
- if also $f'(a) = 0$, then it is a stationary point of inflection

Implicit Differentiation

Used for differentiating circles etc.
If p and q are expressions in x and y such that $p = q$, for all x and y, then:

$$\frac{dp}{dx} = \frac{dq}{dx} \quad \text{and} \quad \frac{dp}{dy} = \frac{dq}{dy}$$

On CAS:

Action → Calculation → `impDiff(y^2+ax=5, x, y)`
Returns $y' = \ldots$.

Integration

$$\int f(x) \cdot dx = F(x) + c \quad \text{where } F'(x) = f(x)$$

Integral laws

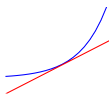
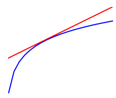
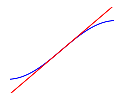
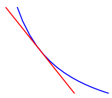
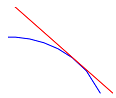
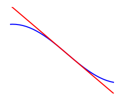
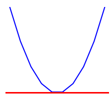
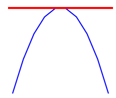
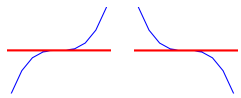
$f(x)$	$\int f(x) \cdot dx$
k (constant)	$kx + c$
x^n	$\frac{1}{n+1}x^{n+1}$
ax^{-n}	$a \cdot \log_e \lvert x \rvert + c$
$\frac{1}{ax+b}$	$\frac{1}{a}\log_e(ax+b) + c$
$(ax+b)^n$	$\frac{1}{a(n+1)}(ax+b)^{n-1} + c \mid n \neq 1$
$(ax+b)^{-1}$	$\frac{1}{a}\log_e \lvert ax+b \rvert + c$
e^{kx}	$\frac{1}{k}e^{kx} + c$
e^k	$e^k x + c$
$\sin kx$	$\frac{-1}{k}\cos(kx) + c$
$\cos kx$	$\frac{1}{k}\sin(kx) + c$
$\sec^2 kx$	$\frac{1}{k}\tan(kx) + c$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a} + c \mid a > 0$
$\frac{-1}{\sqrt{a^2-x^2}}$	$\cos^{-1}\frac{x}{a} + c \mid a > 0$
$\frac{a}{a^2-x^2}$	$\tan^{-1}\frac{x}{a} + c$
$\frac{f'(x)}{f(x)}$	$\log_e f(x) + c$
$\int f(u) \cdot \frac{du}{dx} \cdot dx$	$\int f(u) \cdot du$ (substitution)
$f(x) \cdot g(x)$	$\int [f'(x) \cdot g(x)]dx + \int [g'(x)f(x)]dx$

Note $\sin^{-1}\frac{x}{a} + \cos^{-1}\frac{x}{a}$ is constant $\forall x \in (-a, a)$

Definite integrals

$$\int_a^b f(x) \cdot dx = [F(x)]_a^b = F(b) - F(a)$$

- Signed area enclosed by
  $y = f(x), \quad y = 0, \quad x = a, \quad x = b.$
- *Integrand* is f.

	$\frac{d^2y}{dx^2} > 0$	$\frac{d^2y}{dx^2} < 0$	$\frac{d^2y}{dx^2} = 0$ (inflection)
$\frac{dy}{dx} > 0$	 Rising (concave up)	 Rising (concave down)	 Rising inflection point
$\frac{dy}{dx} < 0$	 Falling (concave up)	 Falling (concave down)	 Falling inflection point
$\frac{dy}{dx} = 0$	 Local minimum	 Local maximum	 Stationary inflection point

Properties

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b k \cdot f(x)\,dx = k\int_a^b f(x)\,dx$$

$$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Integration by substitution

$$\int f(u)\frac{du}{dx} \cdot dx = \int f(u) \cdot du$$

Note $f(u)$ must be 1:1 $\implies$ one x for each y

e.g. for $y = \int (2x+1)\sqrt{x+4} \cdot dx$

let $u = x + 4$

$\implies \frac{du}{dx} = 1$

$\implies x = u - 4$

then $y = \int (2(u-4)+1)u^{\frac{1}{2}} \cdot du$

(solve as normal integral)

Definite integrals by substitution

For $\int_a^b f(x)\frac{du}{dx} \cdot dx$, evaluate new a and b for $f(u) \cdot du$.

Trigonometric integration

$$\sin^m x \cos^n x \cdot dx$$

m **is odd:** $m = 2k+1$ where $k \in \mathbb{Z}$

$\implies \sin^{2k+1} x = (\sin^2 z)^k \sin x = (1-\cos^2 x)^k \sin x$

Substitute $u = \cos x$

n **is odd:** $n = 2k+1$ where $k \in \mathbb{Z}$

$\implies \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1-\sin^2 x)^k \cos x$

Substitute $u = \sin x$

m **and** n **are even:** use identities...

- $\sin^2 x = \frac{1}{2}(1-\cos 2x)$
- $\cos^2 x = \frac{1}{2}(1+\cos 2x)$
- $\sin 2x = 2\sin x \cos x$

Partial fractions

On CAS:

Action → Transformation → `expand/combine`

Interactive → Transformation → Expand → Partial

Graphing integrals on CAS

In main: Interactive → Calculation → $\int$ (→ Definite)

Restrictions: `Define f(x)=..` then `f(x)|x>..`

Applications of antidifferentiation

- x-intercepts of $y = f(x)$ identify x-coordinates of stationary points on $y = F(x)$
- nature of stationary points is determined by sign of $y = f(x)$ on either side of its x-intercepts
- if $f(x)$ is a polynomial of degree n, then $F(x)$ has degree $n + 1$

To find stationary points of a function, substitute x value of given point into derivative. Solve for $\frac{dy}{dx} = 0$. Integrate to find original function.

Solids of revolution

Approximate as sum of infinitesimally-thick cylinders

Rotation about x-axis

$$V = \int_{x-a}^{x=b} \pi y^2 \, dx$$
$$= \pi \int_a^b (f(x))^2 \, dx$$

Rotation about y-axis

$$V = \int_{y=a}^{y=b} \pi x^2 \, dy$$
$$= \pi \int_a^b (f(y))^2 \, dy$$

Regions not bound by $y = 0$

$$V = \pi \int_a^b f(x)^2 - g(x)^2 \, dx$$

where $f(x) > g(x)$

Length of a curve

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} \, dx \quad \text{(Cartesian)}$$

$$L = \int_a^b \sqrt{\frac{dx}{dt} + (\frac{dy}{dt})^2} \, dt \quad \text{(parametric)}$$

On CAS:

Evaluate formula,

or Interactive → Calculation → Line → `arcLen`

Rates

Gradient at a point on parametric curve

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} \mid \frac{dx}{dt} \neq 0 \text{ (chain rule)}$$

$$\frac{d^2}{dx^2} = \frac{d(y')}{dx} = \frac{dy'}{dt} \div \frac{dx}{dt} \mid y' = \frac{dy}{dx}$$

Rational functions

$$f(x) = \frac{P(x)}{Q(x)} \quad \text{where } P, Q \text{ are polynomial functions}$$

Addition of ordinates

- when two graphs have the same ordinate, y-coordinate is double the ordinate
- when two graphs have opposite ordinates, y-coordinate is 0 i.e. (x-intercept)
- when one of the ordinates is 0, the resulting ordinate is equal to the other ordinate

Fundamental theorem of calculus

If f is continuous on $[a, b]$, then

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

where $F = \int f \, dx$

Differential equations

Order - highest power inside derivative

Degree - highest power of highest derivative

e.g. $\left(\frac{dy^2}{d^2}x\right)^3$ order 2, degree 3

Verifying solutions

Start with $y = \ldots$, and differentiate. Substitute into original equation.

Function of the dependent variable

If $\frac{dy}{dx} = g(y)$, then $\frac{dx}{dy} = 1 \div \frac{dy}{dx} = \frac{1}{g(y)}$. Integrate both sides to solve equation. Only add c on one side. Express e^c as A.

Mixing problems

$$\left(\frac{dm}{dt}\right)_\Sigma = \left(\frac{dm}{dt}\right)_\text{in} - \left(\frac{dm}{dt}_\text{out}\right)$$

Separation of variables

If $\frac{dy}{dx} = f(x)g(y)$, then:

$$\int f(x)\,dx = \int \frac{1}{g(y)}\,dy$$

Euler's method for solving DEs

$$\frac{f(x+h)-f(x)}{h} \approx f'(x) \quad \text{for small } h$$

$$\implies f(x+h) \approx f(x) + hf'(x)$$

5 Kinematics & Mechanics

Constant acceleration

- **Position** - relative to origin
- **Displacement** - relative to starting point

Velocity-time graphs

- Displacement: *signed* area between graph and t axis
- Distance travelled: *total* area between graph and t axis

$$\text{acceleration} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

	no
$v = u + at$	x
$v^2 = u^2 + 2as$	t
$s = \frac{1}{2}(v+u)t$	a
$s = ut + \frac{1}{2}at^2$	v
$s = vt - \frac{1}{2}at^2$	u

$$v_\text{avg} = \frac{\Delta\text{position}}{\Delta t}$$

$$\begin{aligned}\text{speed} &= |\text{velocity}| \\ &= \sqrt{v_x^2 + v_y^2 + v_z^2}\end{aligned}$$

Distance travelled between $t = a \to t = b$:

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$$

Shortest distance between $\boldsymbol{r}(t_0)$ and $\boldsymbol{r}(t_1)$:

$$= |\boldsymbol{r}(t_1) - \boldsymbol{r}(t_2)|$$

Vector functions

$$\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

- If $\boldsymbol{r}(t) \equiv$ position with time, then the graph of endpoints of $\boldsymbol{r}(t) \equiv$ Cartesian path
- Domain of $\boldsymbol{r}(t)$ is the range of $x(t)$
- Range of $\boldsymbol{r}(t)$ is the range of $y(t)$

Vector calculus

Derivative

Let $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)(j)$. If both $x(t)$ and $y(t)$ are differentiable, then:

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$$

6 Dynamics

Resolution of forces

Resultant force is sum of force vectors

In angle-magnitude form

$$\text{Cosine rule:} \quad c^2 = a^2 + b^2 - 2ab\cos\theta$$

$$\text{Sine rule:} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

In $\boldsymbol{i}$—$\boldsymbol{j}$ form

Vector of a N at θ to x axis is equal to $a\cos\theta\boldsymbol{i} + a\sin\theta\boldsymbol{j}$.
Convert all force vectors then add.
To find angle of an $a\boldsymbol{i} + b\boldsymbol{j}$ vector, use $\theta = \tan^{-1}\frac{b}{a}$

Resolving in a given direction

The resolved part of a force P at angle θ is has magnitude $P\cos\theta$

To convert force $||\vec{OA}$ to angle-magnitude form, find component $\perp \vec{OA}$ then:

$$|\boldsymbol{r}| = \sqrt{\left(||\vec{OA}\right)^2 + \left(\perp \vec{OA}\right)^2}$$

$$\theta = \tan^{-1}\frac{\perp \vec{OA}}{||\vec{OA}}$$

Newton's laws

1. Velocity is constant without ΣF
2. $\frac{d}{dt}\rho \propto \Sigma F \implies \boldsymbol{F} = m\boldsymbol{a}$
3. Equal and opposite forces

Weight

A mass of m kg has force of mg acting on it

Momentum ρ

$$\rho = mv \qquad \text{(units kg m/s or Ns)}$$

Reaction force R

- With no vertical velocity, $R = mg$
- With vertical acceleration, $|R| = m|a| - mg$
- With force F at angle θ, then $R = mg - F\sin\theta$

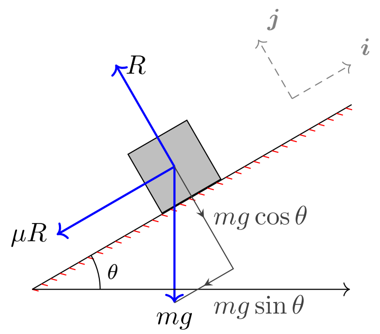
Friction

$$F_R = \mu R \qquad \text{(friction coefficient)}$$

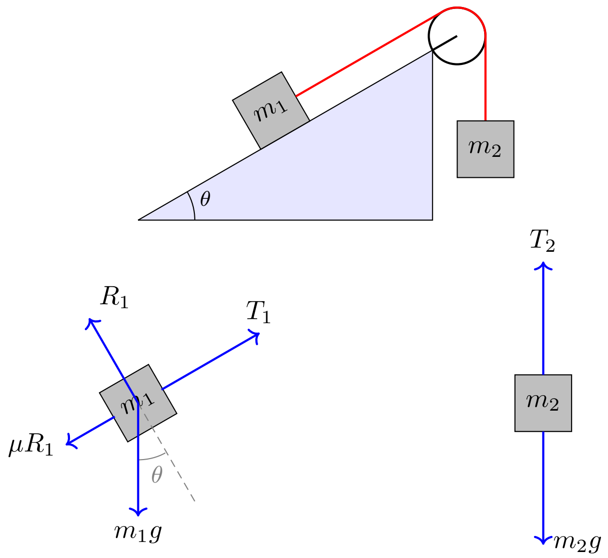
Inclined planes

$$\boldsymbol{F} = |\boldsymbol{F}|\cos\theta\boldsymbol{i} + |\boldsymbol{F}|\sin\theta\boldsymbol{j}$$

- Normal force R is at right angles to plane
- Let direction up the plane be $\boldsymbol{i}$ and perpendicular to plane $\boldsymbol{j}$

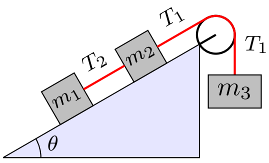


Connected particles



- **Suspended pulley:** tension in both sections of rope are equal
  $|a| = g\frac{m_1 - m_2}{m_1 + m_2}$ where m_1 accelerates down
  With tension:
  $$\begin{cases} m_1 g - T = m_1 a \\ T - m_2 g = m_2 a \end{cases} \implies m_1 g - m_2 g = m_1 a + m_2 a$$
- **String pulling mass on inclined pane:** Resolve parallel to plane
  $$T - mg\sin\theta = ma$$
- **Linear connection:** find acceleration of system first
- **Pulley on right angle:** $a = \frac{m_2 g}{m_1 + m_2}$ where m_2 is suspended (frictionless on both surfaces)
- **Pulley on edge of incline:** find downwards force W_2 and components of mass on plane

In this example, note $T_1 \neq T_2$:



Equilibrium

$$\frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c} \qquad \text{(Lami's theorem)}$$

$$c^2 = a^2 + b^2 - 2ab\cos\theta \qquad \text{(cosine rule)}$$

Three methods:

1. Lami's theorem (sine rule)
2. Triangle of forces (cosine rule)
3. Resolution of forces ($\Sigma F = 0$ - simultaneous)

On CAS

To verify: Geometry tab, then select points with normal cursor. Click right arrow at end of toolbar and input point, then lock known constants.

Variable forces (DEs)

$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

7 Statistics

Continuous random variables

A continuous random variable X has a pdf f such that:

1. $f(x) \geq 0 \forall x$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$E(X) = \int_{\mathbf{X}} (x \cdot f(x))\,dx$$
$$\operatorname{Var}(X) = E\left[(X-\mu)^2\right]$$

$$\Pr(X \leq c) = \int_{-\infty}^{c} f(x)\,dx$$

Two random variables X, Y

If X and Y are independent:

$$\mathrm{E}(aX + bY) = a\,\mathrm{E}(X) + b\,\mathrm{E}(Y)$$
$$\operatorname{Var}(aX \pm bY \pm c) = a^2\operatorname{Var}(X) + b^2\operatorname{Var}(Y)$$

Linear functions $X \to aX + b$

$$\begin{aligned}\Pr(Y \leq y) &= \Pr(aX + b \leq y)\\ &= \Pr\left(X \leq \frac{y-b}{a}\right)\\ &= \int_{-\infty}^{\frac{y-b}{a}} f(x)\,dx\end{aligned}$$

Mean: $\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b$

Variance: $\operatorname{Var}(aX + b) = a^2\operatorname{Var}(X)$

Expectation theorems

For some non-linear function g, the expected value $E(g(X))$ is not equal to $g(E(X))$.

$$E(X^2) = \operatorname{Var}(X) - [E(X)]^2$$
$$E(X^n) = \Sigma x^n \cdot p(x) \qquad \text{(non-linear)}$$
$$\neq [E(X)]^n$$
$$E(aX \pm b) = aE(X) \pm b \qquad \text{(linear)}$$
$$E(b) = b \qquad (\forall b \in \mathbb{R})$$
$$E(X+Y) = E(X) + E(Y) \qquad \text{(two variables)}$$

Sample mean

Approximation of the **population mean** determined experimentally.

$$\overline{x} = \frac{\Sigma x}{n}$$

where

n is the size of the sample (number of sample points)

x is the value of a sample point

On CAS

1. Spreadsheet
2. In cell A1:
   `mean(randNorm(sd, mean, sample size))`
3. Edit → Fill → Fill Range
4. Input range as A1:An where n is the number of samples
5. Graph → Histogram

Sample size of n

$$\overline{X} = \sum_{i=1}^{n} \frac{x_i}{n} = \frac{\sum x}{n}$$

Sample mean is distributed with mean μ and sd $\frac{\sigma}{\sqrt{n}}$ (approaches these values for increasing sample size n).

For a new distribution with mean of n trials, $\mathrm{E}(X') = \mathrm{E}(X)$, $\quad \mathrm{sd}(X') = \dfrac{\mathrm{sd}(X)}{\sqrt{n}}$

On CAS

- Spreadsheet → Catalog → `randNorm(sd, mean, n)` where `n` is the number of samples. Show histogram with Histogram key in top left
- To calculate parameters of a dataset: Calc → One-variable

Normal distributions

$$Z = \frac{X - \mu}{\sigma}$$

Normal distributions must have area (total prob.) of 1 $\implies \int_{-\infty}^{\infty} f(x)\,dx = 1$

mean = mode = median

> Always express z as +ve. Express confidence interval as ordered pair.

Central limit theorem

If X is randomly distributed with mean μ and sd σ, then with an adequate sample size n the distribution of the sample mean $\overline{X}$ is approximately normal with mean $E(\overline{X})$ and $\mathrm{sd}(\overline{X}) = \frac{\sigma}{\sqrt{n}}$.

Confidence intervals

- **Point estimate:** single-valued estimate of the population mean from the value of the sample mean $\overline{x}$
- **Interval estimate:** confidence interval for population mean μ
- $C\%$ confidence interval $\implies$ $C\%$ of samples will contain population mean μ

95% confidence interval

For 95% c.i. of population mean μ:

$$x \in \left(\overline{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}\right)$$

where:

$\overline{x}$ is the sample mean

σ is the population sd

n is the sample size from which $\overline{x}$ was calculated

On CAS

Menu → Stats → Calc → Interval

Set *Type* = *One-Sample Z Int*

and select *Variable*

Margin of error

For 95% confidence interval of μ:

$$M = 1.96 \times \frac{\sigma}{\sqrt{n}}$$

$$\implies n = \left(\frac{1.96\sigma}{M}\right)^2$$

Always round n up to a whole number of samples.

General case

For $C\%$ c.i. of population mean μ:

$$x \in \left(\overline{x} \pm k \frac{\sigma}{\sqrt{n}}\right)$$

where k is such that $\Pr(-k < Z < k) = \frac{C}{100}$

Confidence interval for multiple trials

For a set of n confidence intervals (samples), there is 0.95^n chance that all n intervals contain the population mean μ.

8 Hypothesis testing

Note hypotheses are always expressed in terms of population parameters

Null hypothesis H_0

Sample drawn from population has same mean as control population, and any difference can be explained by sample variations.

Alternative hypothesis H_1

Amount of variation from control is significant, despite standard sample variations.

p-value

$$p = \Pr(\overline{X} \lessgtr \mu(H_1))$$
$$= 2 \cdot \Pr(\overline{X} <> \mu(H_1) | \mu = 8)$$

Probability of observing a value of the sample statistic as significant as the one observed, assuming null hypothesis is true.

p	Conclusion
> 0.05	insufficient evidence against H_0
< 0.05 (5%)	good evidence against H_0
< 0.01 (1%)	strong evidence against H_0
< 0.001 (0.1%)	very strong evidence against H_0

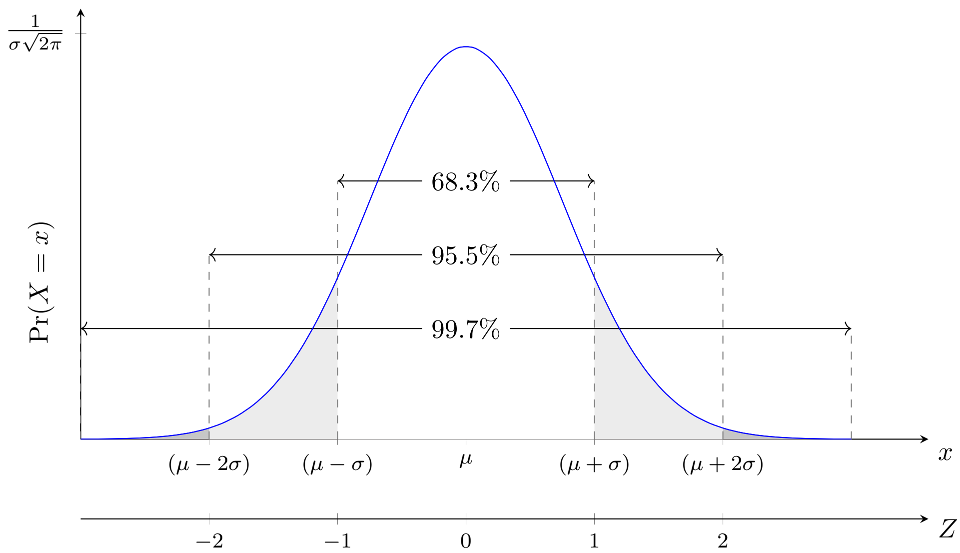
Statistical significance

Significance level is denoted by α.

If $p < \alpha$, null hypothesis is **rejected**
If $p > \alpha$, null hypothesis is **accepted**

z-test

Hypothesis test for a mean of a sample drawn from a normally distributed population with a known standard deviation.



> **On CAS**
>
> Menu → Statistics → Calc → Test.
>
> Select *One-Sample Z-Test* and *Variable*, then input:
>
> | | |
> |---|---|
> | μ cond: | same operator as H_1 |
> | μ_0: | expected sample mean (null hypothesis) |
> | σ: | standard deviation (null hypothesis) |
> | $\overline{x}$: | sample mean |
> | n: | sample size |

One-tail and two-tail tests

One tail

- μ has changed in one direction
- State "$H_1 : \mu \lesseqgtr$ known population mean"

Two tail

- Direction of $\Delta\mu$ is ambiguous
- State "$H_1 : \mu \neq$ known population mean"

For two tail tests:

$$\begin{aligned} p\text{-value} &= \Pr(|\overline{X} - \mu| \geq |\overline{x}_0 - \mu|) \\ &= \left(|Z| \geq \left|\frac{\overline{x}_0 - \mu}{\sigma \div \sqrt{n}}\right|\right) \end{aligned}$$

Modulus notation for two tail

$\Pr(|\overline{X} - \mu| \geq a) \implies$ "the probability that the distance between $\overline{\mu}$ and μ is $\geq a$"

Inverse normal

> **On CAS**
>
> `invNormCdf("L",` α, $\dfrac{\sigma}{n^{\alpha}}$, μ`)`

Errors

Type I error	H_0 is rejected when it is **true**
Type II error	H_0 is **not** rejected when it is **false**