1 Complex numbers

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

Operations

$$\begin{aligned} z_1 \pm z_2 &= (a \pm c)(b \pm d)i \\ k \times z &= ka + kbi \\ z_1 \cdot z_2 &= ac - bd + (ad + bc)i \\ z_1 \div z_2 &= (z_1 \overline{z_2}) \div |z_2|^2 \end{aligned}$$

Conjugate

$$\overline{z} = a \pm bi$$

Properties

$$\begin{aligned} \overline{z_1 \pm z_2} &= \overline{z_1} \pm \overline{z_2} \\ \overline{z_1 \cdot z_2} &= \overline{z_1} \cdot \overline{z_2} \\ \overline{kz} &= k\overline{z} \quad | \quad k \in \mathbb{R} \\ z\overline{z} &= (a + bi)(a - bi) \\ &= a^2 + b^2 \\ &= |z|^2 \end{aligned}$$

Modulus

$$|z| = |\vec{Oz}| = \sqrt{a^2 + b^2}$$

Properties

$$\begin{aligned} |z_1 z_2| &= |z_1||z_2| \\ \left|\frac{z_1}{z_2}\right| &= \frac{|z_1|}{|z_2|} \\ |z_1 + z_2| &\le |z_1| + |z_2| \end{aligned}$$

Multiplicative inverse

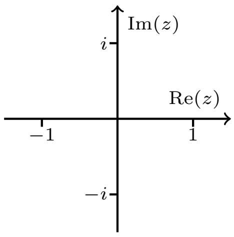
$$\begin{aligned} z^{-1} &= \frac{a - bi}{a^2 + b^2} \\ &= \frac{\overline{z}}{|z|^2} a \end{aligned}$$

Dividing over $\mathbb{C}$

$$\begin{aligned} \frac{z_1}{z_2} &= z_1 z_2^{-1} \\ &= \frac{z_1 \overline{z_2}}{|z_2|^2} \\ &= \frac{(a + bi)(c - di)}{c^2 + d^2} \end{aligned}$$

(rationalise denominator)

Argand planes



Multiplication by $i \implies$ anticlockwise rotation of $\frac{\pi}{2}$

de Moivres' theorem

$$(r \operatorname{cis} \theta)^n = r^n \operatorname{cis}(n\theta) \text{ where } n \in \mathbb{Z}$$

Complex polynomials

Include $\pm$ for all solutions, incl. imaginary

Sum of squares	$z^2 + a^2 = z^2 - (ai)^2$ $= (z + ai)(z - ai)$
Sum of cubes	$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
Division	$P(z) = D(z)Q(z) + R(z)$
Remainder	Let $\alpha \in \mathbb{C}$. Remainder of $P(z) \div (z - \alpha)$ is $P(\alpha)$

Roots

nth roots of $z = r \operatorname{cis} \theta$ are:

$$z = r^{\frac{1}{n}} \operatorname{cis}\left(\frac{\theta + 2k\pi}{n}\right)$$

- Same modulus for all solutions
- Arguments are separated by $\frac{2\pi}{n}$
- Solutions of $z^n = a$ where $a \in \mathbb{C}$ lie on the circle $x^2 + y^2 = \left(|a|^{\frac{1}{n}}\right)^2$

Conjugate root theorem

If $a + bi$ is a solution to $P(z) = 0$, then the conjugate $\overline{z} = a - bi$ is also a solution.