Year 12 Specialist

Andrew Lorimer

2019

Complex & Imaginary Numbers

Imaginary numbers

$$i^2 = -1 \quad \therefore i = \sqrt{-1}$$

Simplifying negative surds

$$\begin{aligned}\sqrt{-2} &= \sqrt{-1 \times 2} \\ &= \sqrt{2}i\end{aligned}$$

Complex numbers

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

General form: $z = a + bi$
$\text{Re}(z) = a, \quad \text{Im}(z) = b$

Addition

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 + z_2 = (a + c) + (b + d)i$$

Subtraction

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 - z_2 = (ac) + (bd)i$$

Multiplication by a real constant

If $z = a + bi$ and $k \in \mathbb{R}$, then

$$kz = ka + kbi$$

Powers of i

- $i^{4n} = 1$
- $i^{4n+1} = i$
- $i^{4n+2} = -1$
- $i^{4n+3} = -i$

For i^n, find remainder r when $n \div 4$. Then $i^n = i^r$.

Multiplying complex expressions

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 \times z_2 = (ac - bd) + (ad + bc)i$$

Conjugates

$$\overline{z} = a \mp bi$$

Properties

- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$
- $\overline{kz} = k\overline{z},\ \text{for } k \in \mathbb{R}$
- $z\overline{z} == (a + bi)(a - bi) = a^2 + b^2 = |z|^2$
- $z + \overline{z} = 2\,\mathrm{Re}(z)$

Modulus

Distance from origin.

$$|z| = \sqrt{a^2 + b^2} \qquad \therefore z\overline{z} = |z|^2$$

Properties

- $|z_1 z_2| = |z_1||z_2|$
- $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}$
- $|z_1 + z_2| \leq |z_1 + |z_2|$

Multiplicative inverse

$$\begin{aligned} z^{-1} &= \frac{1}{z} \\ &= \frac{a - bi}{a^2 + B^2} \\ &= \frac{\overline{z}}{|z|^2} \end{aligned}$$

Dividing complex numbers

$$\frac{z_1}{z_2} = z_1 \ z_2{}^{-1} = \frac{z_1 \overline{z_2}}{|z_2|^2} \quad \text{(multiplicative inverse)}$$

In practice, rationalise denominator:

$$\frac{z_1}{z_2} = \frac{(a + bi)(c - di)}{c^2 + d^2}$$

Argand planes

- Geometric representation of $\mathbb{C}$
- horizontal $= \text{Re}(z)$; vertical $= \text{Im}(z)$
- Multiplication by i results in an anticlockwise rotation of $\frac{\pi}{2}$

Complex polynomials

Include $\pm$ for all solutions, including imaginary

Sum of two squares (quadratics)

$$z^2 + a^2 = z^2 - (ai)^2 = (z + ai)(z - ai)$$

Complete the square to get to this point.

Dividing complex polynomials

$P(z) \div D(z)$ gives quotient $Q(z)$ and remainder $R(z)$:

$$P(z) = D(z)Q(z) + R(z)$$

Remainder theorem

Let $\alpha \in \mathbb{C}$. Remainder of $P(z) \div (z - \alpha)$ is $P(\alpha)$

Factor theorem

If $a + bi$ is a solution to $P(z) = 0$, then:

- $P(a + bi) = 0$
- $z - (a + bi)$ is a factor of $P(z)$

Sum of two cubes

$$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$$

Conjugate root theorem

If $a+bi$ is a solution to $P(z) = 0$, then the conjugate $\overline{z} = a-bi$ is also a solution.

Polar form

$$\begin{aligned} z &= r \operatorname{cis} \theta \\ &= r(\cos\theta + i \sin\theta) \\ &= a + bi \end{aligned}$$

- $r = |z| = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}$
- $\theta = \arg(z)$ (on CAS: `arg(a+bi)`)
- **principal argument** is $\operatorname{Arg}(z) \in (-\pi, \pi]$ (note capital Arg)

Each complex number has multiple polar representations:
$z = r \operatorname{cis}\theta = r \operatorname{cis}(\theta + 2n\pi)$ with $n \in \mathbb{Z}$ revolutions

Conjugate in polar form

$$(r \operatorname{cis}\theta)^{-1} = r \operatorname{cis}(-\theta)$$

Reflection of z across horizontal axis.

Multiplication and division in polar form

$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

de Moivres' Theorem

$$(r \operatorname{cis}\theta)^n = r^n \operatorname{cis}(n\theta) \text{ where } n \in \mathbb{Z}$$

Roots of complex numbers

nth roots of $z = r \operatorname{cis}\theta$ are

$$z = r^{\frac{1}{n}} \operatorname{cis}(\frac{\theta + 2k\pi}{n})$$

Same modulus for all solutions. Arguments are separated by $\frac{2\pi}{n}$

The solutions of $z^n = a$ where $a \in \mathbb{C}$ lie on circle

$$x^2 + y^2 = (|a|^{\frac{1}{n}})^2$$

Sketching complex graphs

Straight line

- $\text{Re}(z) = c$ or $\text{Im}(z) = c$ (perpendicular bisector)
- $\text{Arg}(z) = \theta$
- $|z + a| = |z + bi|$ where $m = \frac{a}{b}$

- $|z+a| = |z+b| \longrightarrow 2(a-b)x = b^2 - a^2$

Circle

$|z - z_1|^2 = c^2|z_2 + 2|^2$ or $|z - (a+bi)| = c$

Locus

$\text{Arg}(z) < \theta$

Vectors

- **vector:** a directed line segment

- arrow indicates direction
- length indicates magnitude
- column notation: $\begin{bmatrix} x \\ y \end{bmatrix}$
- vectors with equal magnitude and direction are equivalent

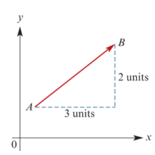


Figure 1:

Vector addition

$\boldsymbol{u} + \boldsymbol{v}$ can be represented by drawing each vector head to tail then joining the lines.
Addition is commutative (parallelogram)

Scalar multiplication

For $k \in \mathbb{R}^+$, $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$ but length is multiplied by a factor of k.

When multiplied by $k < 0$, direction is reversed and length is multplied by k.

Vector subtraction

To find $\boldsymbol{u} - \boldsymbol{v}$, add $\boldsymbol{-v}$ to $\boldsymbol{u}$

Parallel vectors

Same or opposite direction

$$\boldsymbol{u} || \boldsymbol{v} \iff \boldsymbol{u} = k\boldsymbol{v} \text{ where } k \in \mathbb{R} \setminus \{0\}$$

Position vectors

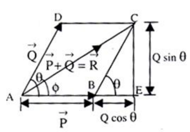
Vectors may describe a position relative to O.

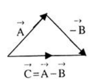
For a point A, the position vector is

Linear combinations of non-parallel vectors

If two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel, then:

$$m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b} \quad \therefore \quad m = p,\ n = q$$





Column vector notation

A vector between points $A(x_1, y_1),\ B(x_2, y_2)$ can be represented as $\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$

Component notation

A vector $\boldsymbol{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ can be written as $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$.
$\boldsymbol{u}$ is the sum of two components $x\boldsymbol{i}$ and $y\boldsymbol{j}$
Magnitude of vector $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$ is denoted by $|u| = \sqrt{x^2 + y^2}$

Basic algebra applies:
$(x\boldsymbol{i} + y\boldsymbol{j}) + (m\boldsymbol{i} + n\boldsymbol{j}) = (x + m)\boldsymbol{i} + (y + n)\boldsymbol{j}$
Two vectors equal if and only if their components are equal.

Unit vector \hat{\boldsymbol{a}}=1

$$\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$$
$$= \boldsymbol{a} \cdot |\boldsymbol{a}|$$

Scalar/dot product \boldsymbol{a} \cdot \boldsymbol{b}

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$

on CAS: `dotP([a b c], [d e f])`

Scalar product properties

1. $k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (kb)$
2. $\boldsymbol{a} \cdot \boldsymbol{0} = 0$
3. $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
4. $\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1$
5. If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular
6. $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2 = a^2$

For parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{cases} |\boldsymbol{a}||\boldsymbol{b}| & \text{if same direction} \\ -|\boldsymbol{a}||\boldsymbol{b}| & \text{if opposite directions} \end{cases}$$

Geometric scalar products

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$$

where $0 \leq \theta \leq \pi$

Perpendicular vectors

If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, then $\boldsymbol{a} \perp \boldsymbol{b}$ (since $\cos 90 = 0$)

Finding angle between vectors

positive direction

$$\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{a_1b_1 + a_2b_2}{|\boldsymbol{a}||\boldsymbol{b}|}$$

on CAS: `angle([a b c], [a b c])` (Action -> Vector -> Angle)

Angle between vector and axis

Direction of a vector can be given by the angles it makes with $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ directions.

For $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ which makes angles α, β, γ with positive direction of x, y, z axes:

$$\cos\alpha = \frac{a_1}{|\boldsymbol{a}|}, \quad \cos\beta = \frac{a_2}{|\boldsymbol{a}|}, \quad \cos\gamma = \frac{a_3}{|\boldsymbol{a}|}$$

on CAS: `angle([a b c], [1 0 0])` for angle between $a\boldsymbol{i}+b\boldsymbol{j}+c\boldsymbol{k}$ and x-axis

Vector projections

Vector resolute of $\boldsymbol{a}$ in direction of $\boldsymbol{b}$ is magnitude of $\boldsymbol{a}$ in direction of $\boldsymbol{b}$:

$$\boldsymbol{u} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|^2}\boldsymbol{b} = \left(\boldsymbol{a}\cdot\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right)\left(\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right) = (\boldsymbol{a}\cdot\hat{\boldsymbol{b}})\hat{\boldsymbol{b}}$$

Scalar resolute of \boldsymbol{a} on \boldsymbol{b}

$$r_s = |\boldsymbol{u}| = \boldsymbol{a}\cdot\hat{\boldsymbol{b}}$$

Vector resolute of \boldsymbol{a} \perp \boldsymbol{b}

$$\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u} \text{ where } \boldsymbol{u} \text{ is projection } \boldsymbol{a} \text{ on } \boldsymbol{b}$$

Vector proofs

Concurrent lines

$\geq$ 3 lines intersect at a single point

Collinear points

≥ 3 points lie on the same line
$\implies \vec{OC} = \lambda\vec{OA} + \mu\vec{OB}$ where $\lambda + \mu = 1$. If C is between $\vec{AB}$, then $0 < \mu < 1$
Points A, B, C are collinear iff $\vec{AC} = m\vec{AB}$ where $m \neq 0$

Useful vector properties

- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then $\boldsymbol{b} = k\boldsymbol{a}$ for some $k \in \mathbb{R} \setminus \{0\}$
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel with at least one point in common, then they lie on the same straight line
- Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$

Linear dependence

Vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are linearly dependent if they are non-parallel and:

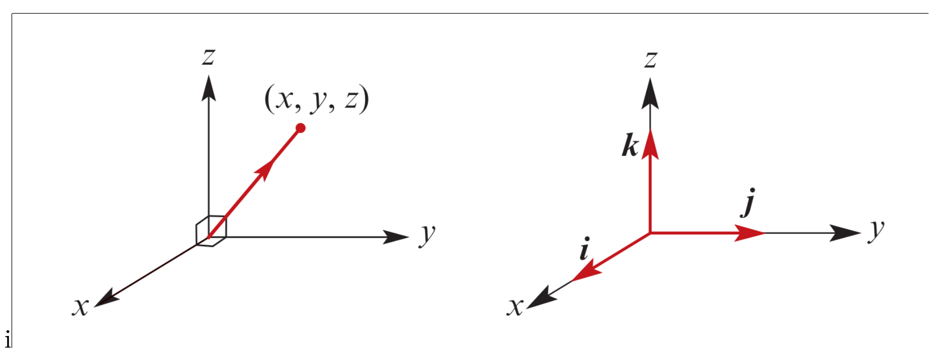
$$k\boldsymbol{a} + l\boldsymbol{b} + m\boldsymbol{c} = 0$$
$$\therefore \boldsymbol{c} = m\boldsymbol{a} + n\boldsymbol{b} \quad \text{(simultaneous)}$$

$\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$ are linearly independent if no vector in the set is expressible as a linear combination of other vectors in set, or if they are parallel.

Vector $\boldsymbol{w}$ is a linear combination of vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$

Three-dimensional vectors

Right-hand rule for axes: z is up or out of page.



Parametric vectors

Parametric equation of line through point (x_0, y_0, z_0) and parallel to $a\boldsymbol{i}+b\boldsymbol{j}+c\boldsymbol{k}$ is:

$$\begin{cases} x = x_o + a \cdot t \\ y = y_0 + b \cdot t \\ z = z_0 + c \cdot t \end{cases}$$

Circular functions

Period of $a\sin(bx)$ is $\frac{2\pi}{b}$

Period of $a\tan(nx)$ is $\frac{\pi}{n}$
Asymptotes at $x = \frac{2k+1)\pi}{2n} \mid k \in \mathbb{Z}$

Reciprocal functions

Cosecant

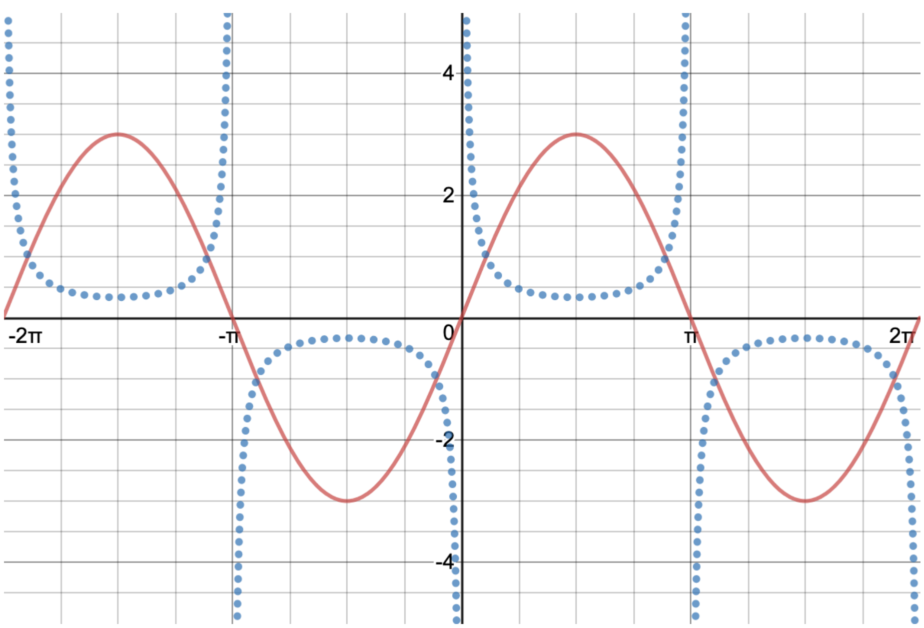


Figure 2:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \mid \sin\theta \neq 0$$

- **Domain** $= \mathbb{R} \setminus n\pi : n \in \mathbb{Z}$
- **Range** $= \mathbb{R} \setminus (-1, 1)$
- **Turning points** at $\theta = \frac{(2n+1)\pi}{2} \mid n \in \mathbb{Z}$
- **Asymptotes** at $\theta = n\pi \mid n \in \mathbb{Z}$

Secant

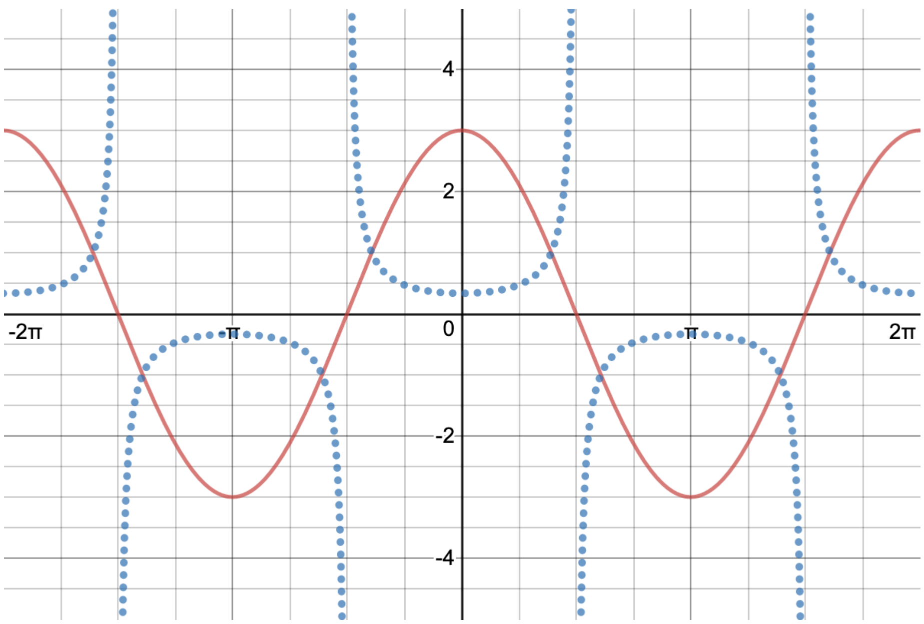


Figure 3:

$$\sec\theta = \frac{1}{\cos\theta} \mid \cos\theta \neq 0$$

- **Domain** $= \mathbb{R} \setminus \{\frac{(2n+1)\pi}{2} : n \in \mathbb{Z}\}$
- **Range** $= \mathbb{R} \setminus (-1, 1)$
- **Turning points** at $\theta = n\pi \mid n \in \mathbb{Z}$
- **Asymptotes** at $\theta = \frac{(2n+1)\pi}{2} \mid n \in \mathbb{Z}$

Cotangent

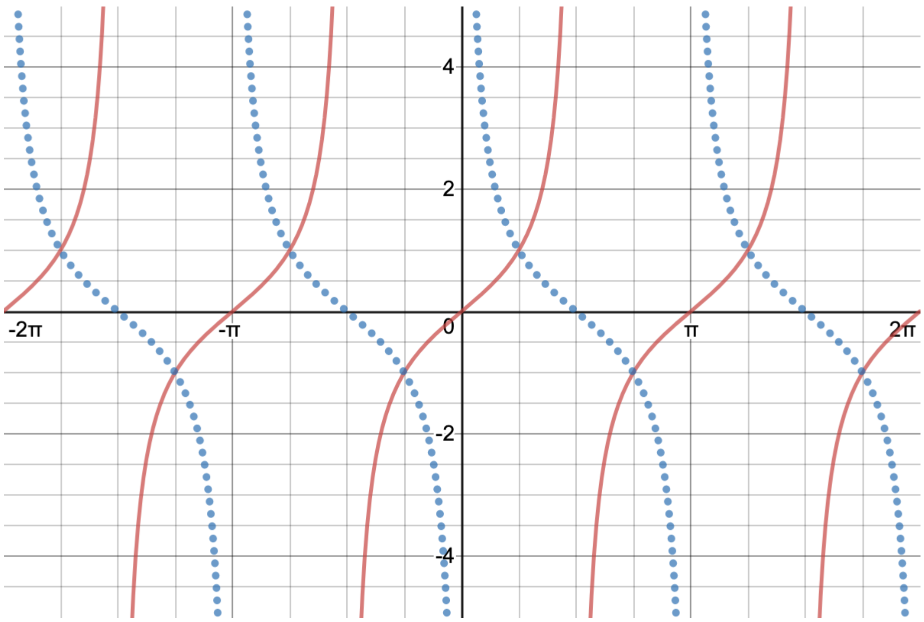


Figure 4:

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \mid \sin\theta \neq 0$$

- **Domain** $= \mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$
- **Range** $= \mathbb{R}$
- **Asymptotes** at $\theta = n\pi \mid n \in \mathbb{Z}$

Symmetry properties

$$\begin{aligned}
\sec(\pi \pm x) &= -\sec x \\
\sec(-x) &= \sec x \\
\operatorname{cosec}(\pi \pm x) &= \mp \operatorname{cosec} x \\
\operatorname{cosec}(-x) &= -\operatorname{cosec} x \\
\cot(\pi \pm x) &= \pm \cot x \\
\cot(-x) &= -\cot x
\end{aligned}$$

Complementary properties

$$\sec\left(\frac{\pi}{2} - x\right) = \operatorname{cosec} x$$
$$\operatorname{cosec}\left(\frac{\pi}{2} - x\right) = \sec x$$
$$\cot\left(\frac{\pi}{2} - x\right) = \tan x$$
$$\tan\left(\frac{\pi}{2} - x\right) = \cot x$$

Pythagorean identities

$$1 + \cot^2 x = \operatorname{cosec}^2 x, \quad \text{where } \sin x \neq 0$$
$$1 + \tan^2 x = \sec^2 x, \quad \text{where } \cos x \neq 0$$

Compound angle formulas

$$\cos(x \pm y) = \cos x + \cos y \mp \sin x \sin y$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

Double angle formulas

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \\ &= 2\cos^2 x - 1 \end{aligned}$$

$$\sin 2x = 2 \sin x \cos x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Inverse circular functions

Inverse functions: $f(f^{-1}(x)) = x, \quad f(f^{-1}(x)) = x$
Must be 1:1 to find inverse (reflection in $y = x$

Domain is restricted to make functions 1:1.

\arcsin

$$\sin^{-1} : [-1, 1] \to \mathbb{R}, \quad \sin^{-1} x = y, \quad \text{where } \sin y = x \text{ and } y \in [\frac{-\pi}{2}, \frac{\pi}{2}]$$

\arcos

$$\cos^{-1} \to \mathbb{R}, \quad \cos^{-1} x = y, \quad \text{where } \cos y = x \text{ and } y \in [0, \pi]$$

\arctan

$$\tan^{-1} : \mathbb{R} \to \mathbb{R}, \quad \tan^{-1} x = y, \quad \text{where } \tan y = x \text{ and } y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Differential calculus

Limits

$$\lim_{x \to a} f(x)$$

L^- - limit from below

L^+ - limit from above

$\lim_{x \to a} f(x)$ - limit of a point

- Limit exists if $L^- = L^+$
- If limit exists, point does not.

Limits can be solved using normal techniques (if div 0, factorise)

Limit theorems

1. For constant function $f(x) = k$, $\lim_{x \to a} f(x) = k$
2. $\lim_{x \to a}(f(x) \pm g(x)) = F \pm G$
3. $\lim_{x \to a}(f(x) \times g(x)) = F \times G$
4. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G}, G \neq 0$

Corollary: $\lim_{x \to a} c \times f(x) = cF$ where $c =$ constant

Solving limits for x\rightarrow\infty

Factorise so that all values of x are in denominators.

e.g.

$$\lim_{x\to\infty} \frac{2x+3}{x-2} = \frac{2+\frac{3}{x}}{1-\frac{2}{x}} = \frac{2}{1} = 2$$

Continuous functions

A function is continuous if $L^- = L^+ = f(x)$ for all values of x.

Gradients of secants and tangents

Secant (chord) - line joining two points on curve

Tangent - line that intersects curve at one point

given $P(x, y) \quad Q(x + \delta x, y + \delta y)$: gradient of chord joining P and Q is $m_{PQ} = \frac{\text{rise}}{\text{run}} = \frac{\delta y}{\delta x}$

As $Q \to P, \delta x \to 0$. Chord becomes tangent (two infinitesimal points are equal).

Can also be used with functions, where $h = \delta x$.

First principles derivative

$$f'(x) = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

$$m_{\tan} = \lim_{h\to 0} f'(x)$$

$$m_{\vec{PQ}} = f'(x)$$

first principles derivative:

$$m_{\text{tangent at } P} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Gradient at a point

Given point $P(a, b)$ and function $f(x)$, the gradient is $f'(a)$

Derivatives of x^n

$$\frac{d(ax^n)}{dx} = anx^{n-1}$$

If $x =$ constant, derivative is 0

If $y = ax^n$, derivative is $a \times nx^{n-1}$

If $f(x) = \frac{1}{x} = x^{-1}, \quad f'(x) = -1x^{-2} = \frac{-1}{x^2}$

If $f(x) =^5 \sqrt{x} = x^{\frac{1}{5}}, \quad f'(x) = \frac{1}{5}x^{-4/5} = \frac{1}{5\times ^5\sqrt{x^4}}$

If $f(x) = (x-b)^2, \quad f'(x) = 2(x-b)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives of u \pm v

$$\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$$

where u and v are functions of x

Euler's number as a limit

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

Chain rule for (f\circ g)

If $f(x) = h(g(x)) = (h \circ g)(x)$:

$$f'(x) = h'(g(x)) \cdot g'(x)$$

If $y = h(u)$ and $u = g(x)$:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{d((ax+b)^n)}{dx} = \frac{d(ax+b)}{dx} \cdot n \cdot (ax+b)^{n-1}$$

Used with only one expression.

e.g. $y = (x^2 + 5)^7$ - Cannot reasonably expand
Let $u - x^2 + 5$ (inner expression)
$\frac{du}{dx} = 2x$
$y = u^7$
$\frac{dy}{du} = 7u^6$

Product rule for y=uv

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient rule for y={u \over v}

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}$$

Logarithms

$$\log_b(x) = n \quad \text{where} \quad b^n = x$$

Wikipedia:

> the logarithm of a given number x is the exponent to which another fixed number, the base b, must be raised, to produce that number x

Logarithmic identities

$\log_b(xy) = \log_b x + \log_b y$
$\log_b x^n = n \log_b x$
$\log_b y^{x^n} = x^n \log_b y$

Index identities

$b^{m+n} = b^m \cdot b^n$
$(b^m)^n = b^{m \cdot n}$
$(b \cdot c)^n = b^n \cdot c^n$
$a^m \div a^n = a^{m-n}$

e as a logarithm

$$\text{if } y = e^x, \quad \text{then } x = \log_e y$$
$$\ln x = \log_e x$$

Differentiating logarithms

$$\frac{d(\log_e x)}{dx} = x^{-1} = \frac{1}{x}$$

Derivative rules

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\sin ax$	$a \cos ax$
$\cos x$	$-\sin x$
$\cos ax$	$-a \sin ax$
$\tan f(x)$	$f^2(x) \sec^2 f(x)$
e^x	e^x
e^{ax}	ae^{ax}
ax^{nx}	$an \cdot e^{nx}$
$\log_e x$	$\frac{1}{x}$
$\log_e ax$	$\frac{1}{x}$
$\log_e f(x)$	$\frac{f'(x)}{f(x)}$
$\sin(f(x))$	$f'(x) \cdot \cos(f(x))$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{sqrt1-x^2}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$

Reciprocal derivatives

$$\frac{1}{\frac{dy}{dx}} = \frac{dx}{dy}$$

Differentiating x=f(y)

Find $\frac{dx}{dy}$. Then $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \implies \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Second derivative

$$f(x) \longrightarrow f'(x) \longrightarrow f''(x)$$

$$\therefore y \longrightarrow \frac{dy}{dx} \longrightarrow \frac{d(\frac{dy}{dx})}{dx} \longrightarrow \frac{d^2y}{dx^2}$$

Order of polynomial nth derivative decrements each time the derivative is taken

Points of Inflection

Stationary point - point of zero gradient (i.e. $f'(x) = 0$)
Point of inflection - point of maximum |gradient| (i.e. $f'' = 0$)

- if $f'(a) = 0$ and $f''(a) > 0$, then point $(a, f(a))$ is a local min (curve is concave up)
- if $f'(a) = 0$ and $f''(a) < 0$, then point $(a, f(a))$ is local max (curve is concave down)
- if $f''(a) = 0$, then point $(a, f(a))$ is a point of inflection
- if also $f'(a) = 0$, then it is a stationary point of inflection

Implicit Differentiation

On CAS: Action → Calculation → `impDiff(y^2+ax=5, x, y)`. Returns $y' = \ldots$.

Used for differentiating circles etc.

If p and q are expressions in x and y such that $p = q$, for all x nd y, then:

$$\frac{dp}{dx} = \frac{dq}{dx} \quad \text{and} \quad \frac{dp}{dy} = \frac{dq}{dy}$$

Integration

$$\int f(x) \cdot dx = F(x) + c \quad \text{where } F'(x) = f(x)$$

$$\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + c$$

- area enclosed by curves
- $+c$ should be shown on each step without $\int$

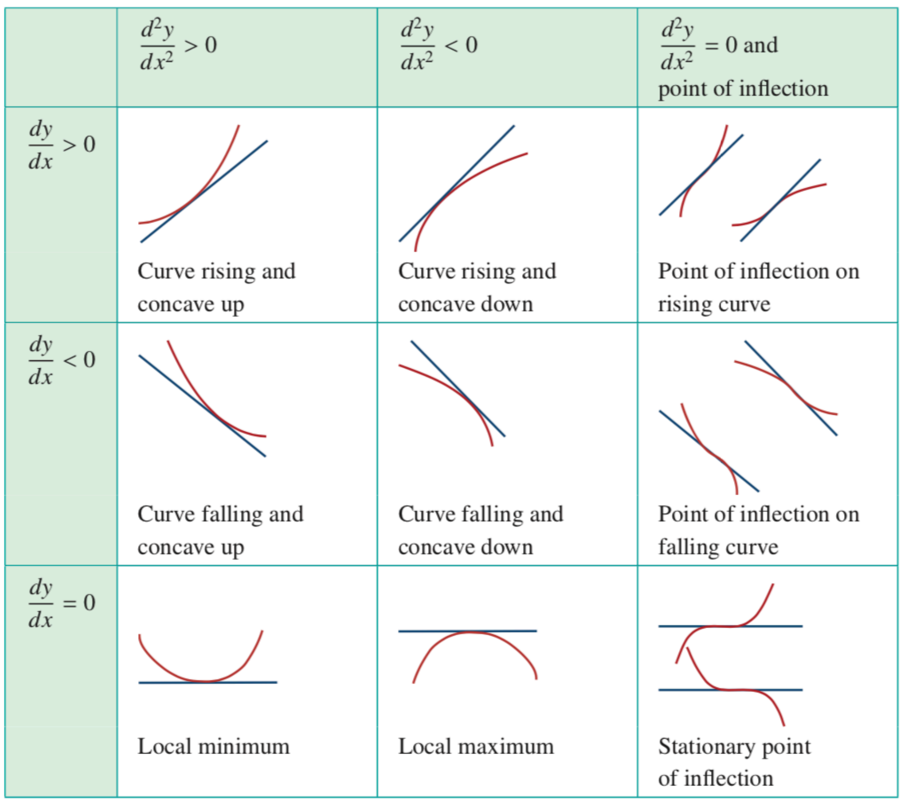

$\frac{d^2y}{dx^2} > 0$
$\frac{d^2y}{dx^2} < 0$
$\frac{d^2y}{dx^2} = 0$ and point of inflection
$\frac{dy}{dx} > 0$
Curve rising and concave up
Curve rising and concave down
Point of inflection on rising curve
$\frac{dy}{dx} < 0$
Curve falling and concave up
Curve falling and concave down
Point of inflection on falling curve
$\frac{dy}{dx} = 0$
Local minimum
Local maximum
Stationary point of inflection

Figure 5:

Integral laws

$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$
$\int kf(x)dx = k \int f(x)dx$

$f(x)$	$\int f(x) \cdot dx$
k (constant)	$kx + c$
x^n	$\frac{x^{n+1}}{n+1} + c$
ax^{-n}	$a \cdot \log_e x + c$
$\frac{1}{ax+b}$	$\frac{1}{a} \log_e(ax + b) + c$
$(ax + b)^n$	$\frac{1}{a(n+1)}(ax + b)^{n-1} + c$
e^{kx}	$\frac{1}{k}e^{kx} + c$
e^k	$e^k x + c$
$\sin kx$	$-\frac{1}{k} \cos(kx) + c$
$\cos kx$	$\frac{1}{k} \sin(kx) + c$
$\sec^2 kx$	$\frac{1}{k} \tan(kx) + c$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1} \frac{x}{a} + c \mid a > 0$
$\frac{-1}{\sqrt{a^2-x^2}}$	$\cos^{-1} \frac{x}{a} + c \mid a > 0$
$\frac{a}{a^2-x^2}$	$\tan^{-1} \frac{x}{a} + c$
$\frac{f'(x)}{f(x)}$	$\log_e f(x) + c$
$g'(x) \cdot f'(g(x)$	$f(g(x))$ (chain rule)
$f(x) \cdot g(x)$	$\int [f'(x) \cdot g(x)]dx + \int [g'(x)f(x)]dx$

Note $\sin^{-1} \frac{x}{a} + \cos^{-1} \frac{x}{a}$ is constant for all $x \in (-a, a)$.

Definite integrals

$$\int_a^b f(x) \cdot dx = [F(x)]_a^b = F(b) - F(a)$$

- Signed area enclosed by: $y = f(x), \quad y = 0, \quad x = a, \quad x = b.$
- *Integrand* is f.
- $F(x)$ may be any integral, i.e. c is inconsequential

Properties

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b k \cdot f(x)\,dx = k\int_a^b f(x)\,dx$$

$$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Integration by substitution

$$\int f(u)\frac{du}{dx} \cdot dx = \int f(u) \cdot du$$

Note $f(u)$ must be one-to-one $\implies$ one x value for each y value

e.g. for $y = \int (2x+1)\sqrt{x+4} \cdot dx$:
let $u = x + 4$
$\implies \frac{du}{dx} = 1$
$\implies x = u - 4$
then $y = \int (2(u-4)+1)u^{\frac{1}{2}} \cdot du$
Solve as a normal integral

Definite integrals by substitution

For $\int_a^b f(x)\frac{du}{dx} \cdot dx$, evaluate new a and b for $f(u) \cdot du$.

Trigonometric integration

$$\sin^m x \cos^n x \cdot dx$$

m is odd:
$m = 2k + 1$ where $k \in \mathbb{Z}$
$\implies \sin^{2k+1} x = (\sin^2 z)^k \sin x = (1 - \cos^2 x)^k \sin x$
Substitute $u = \cos x$

n is odd:
$n = 2k + 1$ where $k \in \mathbb{Z}$
$\implies \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x$
Subbstitute $u = \sin x$

m and n are even:
Use identities:

- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin 2x = 2 \sin x \cos x$

Partial fractions

On CAS: Action → Transformation → `expand/combine`
or Interactive → Transformation → `expand` → Partial

Graphing integrals on CAS

In main: Interactive → Calculation → $\int$ (→ Definite)
Restrictions: `Define f(x)=...` → `f(x)x>1` (e.g.)

Applications of antidifferentiation

- x-intercepts of $y = f(x)$ identify x-coordinates of stationary points on $y = F(x)$
- nature of stationary points is determined by sign of $y = f(x)$ on either side of its x-intercepts
- if $f(x)$ is a polynomial of degree n, then $F(x)$ has degree $n + 1$

To find stationary points of a function, substitute x value of given point into derivative. Solve for $\frac{dy}{dx} = 0$. Integrate to find original function.

Solids of revolution

Approximate as sum of infinitesimally-thick cylinders

Rotation about x-axis

$$
\begin{aligned}
V &= \int_{x-a}^{x=b} \pi y^2 \, dx \\
&= \pi \int_a^b (f(x))^2 \, dx
\end{aligned}
$$

Rotation about y-axis

$$V = \int_{y=a}^{y=b} \pi x^2 \, dy$$
$$= \pi \int_a^b (f(y))^2 \, dy$$

Regions not bound by y=0

$$V = \pi \int_a^b f(x)^2 - g(x)^2 \, dx$$

where $f(x) > g(x)$

Length of a curve

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} \, dx \quad \text{(Cartesian)}$$

$$L = \int_a^b \sqrt{\frac{dx}{dt} + (\frac{dy}{dt})^2} \, dt \quad \text{(parametric)}$$

Evaluate on CAS. Or use Interactive → Calculation → Line → `arcLen`.

Rates

Related rates

$$\frac{da}{db} \quad \text{(change in } a \text{ with respect to } b\text{)}$$

Gradient at a point on parametric curve

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} \mid \frac{dx}{dt} \neq 0$$

$$\frac{d^2}{dx^2} = \frac{d(y')}{dx} = \frac{dy'}{dt} \div \frac{dx}{dt} \mid y' = \frac{dy}{dx}$$

Rational functions

$$f(x) = \frac{P(x)}{Q(x)} \quad \text{where } P, Q \text{ are polynomial functions}$$

Addition of ordinates

- when two graphs have the same ordinate, y-coordinate is double the ordinate
- when two graphs have opposite ordinates, y-coordinate is 0 i.e. (x-intercept)
- when one of the ordinates is 0, the resulting ordinate is equal to the other ordinate

Fundamental theorem of calculus

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f

Differential equations

One or more derivatives

Order - highest power inside derivative
Degree - highest power of highest derivative
e.g. $\left(\frac{dy^2}{d^2x}\right)^3$: order 2, degree 3

Verifying solutions

Start with $y = \ldots$, and differentiate. Substitute into original equation.

Function of the dependent variable

If $\frac{dy}{dx} = g(y)$, then $\frac{dx}{dy} = 1 \div \frac{dy}{dx} = \frac{1}{g(y)}$. Integrate both sides to solve equation. Only add c on one side. Express e^c as A.

Mixing problems

$$\left(\frac{dm}{dt}\right)_{\Sigma} = \left(\frac{dm}{dt}\right)_{\text{in}} - \left(\frac{dm}{dt}\right)_{\text{out}}$$

Separation of variables

If $\frac{dy}{dx} = f(x)g(y)$, then:

$$\int f(x)\,dx = \int \frac{1}{g(y)}\,dy$$

Using definite integrals to solve DEs

Used for situations where solutions to $\frac{dy}{dx} = f(x)$ is not required.

In some cases, it may not be possible to obtain an exact solution.

Approximate solutions can be found by numerically evaluating a definite integral.

Using Euler's method to solve a differential equation

$$\frac{f(x+h) - f(x)}{h} \approx f'(x) \quad \text{for small } h$$

$$\implies f(x+h) \approx f(x) + hf'(x)$$