Year 12 Specialist

Andrew Lorimer

1 Complex numbers

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

Cartesian form: $a + bi$

Polar form: $r \operatorname{cis} \theta$

Operations

	Cartesian	Polar
$z_1 \pm z_2$	$(a \pm c)(b \pm d)i$	convert to $a + bi$
$+k \times z$	$ka \pm kbi$	$kr \operatorname{cis} \theta$
$-k \times z$		$kr \operatorname{cis}(\theta \pm \pi)$
$z_1 \cdot z_2$	$ac - bd + (ad + bc)i$	$r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$
$z_1 \div z_2$	$(z_1 \overline{z_2}) \div \lvert z_2 \rvert^2$	$\left(\frac{r_1}{r_2}\right) \operatorname{cis}(\theta_1 - \theta_2)$

Scalar multiplication in polar form

For $k \in \mathbb{R}^+$:

$$k\,(r \operatorname{cis} \theta) = kr \operatorname{cis} \theta$$

For $k \in \mathbb{R}^-$:

$$k\,(r \operatorname{cis} \theta) = kr \operatorname{cis} \left(\begin{cases} \theta - \pi & |0 < \operatorname{Arg}(z) \le \pi \\ \theta + \pi & |-\pi < \operatorname{Arg}(z) \le 0 \end{cases} \right)$$

Conjugate

$$\begin{aligned} \overline{z} &= a \mp bi \\ &= r \operatorname{cis}(-\theta) \end{aligned}$$

On CAS: `conjg(a+bi)`

Properties

$$\begin{aligned} \overline{z_1 \pm z_2} &= \overline{z_1} \pm \overline{z_2} \\ \overline{z_1 \cdot z_2} &= \overline{z_1} \cdot \overline{z_2} \\ \overline{kz} &= k\overline{z} \,\forall\, k \in \mathbb{R} \\ z\overline{z} &= (a + bi)(a - bi) \\ &= a^2 + b^2 \\ &= |z|^2 \end{aligned}$$

Modulus

$$|z| = |\vec{Oz}| = \sqrt{a^2 + b^2}$$

Properties

$$\begin{aligned} |z_1 z_2| &= |z_1||z_2| \\ \left|\frac{z_1}{z_2}\right| &= \frac{|z_1|}{|z_2|} \\ |z_1 + z_2| &\le |z_1| + |z_2| \end{aligned}$$

Multiplicative inverse

$$\begin{aligned} z^{-1} &= \frac{a - bi}{a^2 + b^2} \\ &= \frac{\overline{z}}{|z|^2} a \\ &= r \operatorname{cis}(-\theta) \end{aligned}$$

Dividing over $\mathbb{C}$

$$\begin{aligned} \frac{z_1}{z_2} &= z_1 z_2^{-1} \\ &= \frac{z_1 \overline{z_2}}{|z_2|^2} \\ &= \frac{(a + bi)(c - di)}{c^2 + d^2} \end{aligned}$$

then rationalise denominator

Polar form

$$\begin{aligned} z &= r \operatorname{cis} \theta \\ &= r(\cos \theta + i \sin \theta) \end{aligned}$$

- $r = |z| = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}$
- $\theta = \arg(z)$ On CAS: `arg(a+bi)`
- $\operatorname{Arg}(z) \in (-\pi, \pi)$ **(principal argument)**
- Multiple representations:
  $r \operatorname{cis} \theta = r \operatorname{cis}(\theta + 2n\pi)$ with $n \in \mathbb{Z}$ revolutions
- $\operatorname{cis} \pi = -1, \quad \operatorname{cis} 0 = 1$

On CAS

`compToTrig(a+bi)` $\iff$ `cExpand{r·cisX}`

de Moivres' theorem

$$(r \operatorname{cis}\theta)^n = r^n \operatorname{cis}(n\theta) \text{ where } n \in \mathbb{Z}$$

Complex polynomials

Include $\pm$ for all solutions, incl. imaginary

Sum of squares	$z^2 + a^2 = z^2 - (ai)^2$ $= (z+ai)(z-ai)$
Sum of cubes	$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
Division	$P(z) = D(z)Q(z) + R(z)$
Remainder theorem	Let $\alpha \in \mathbb{C}$. Remainder of $P(z) \div (z-\alpha)$ is $P(\alpha)$
Factor theorem	$z - \alpha$ is a factor of $P(z) \iff P(\alpha) = 0$ for $\alpha \in \mathbb{C}$
Conjugate root theorem	$P(z) = 0$ at $z = a \pm bi$ ($\implies$ both z_1 and $\overline{z_1}$ are solutions)

nth roots

nth roots of $z = r \operatorname{cis}\theta$ are:

$$z = r^{\frac{1}{n}} \operatorname{cis}\left(\frac{\theta + 2k\pi}{n}\right)$$

- Same modulus for all solutions
- Arguments separated by $\frac{2\pi}{n}$ $\therefore$ there are n roots
- If one square root is $a + bi$, the other is $-a - bi$
- Give one implicit nth root z_1, function is $z = z_1^n$
- Solutions of $z^n = a$ where $a \in \mathbb{C}$ lie on the circle $x^2 + y^2 = \left(|a|^{\frac{1}{n}}\right)^2$ (intervals of $\frac{2\pi}{n}$)

For $0 = az^2 + bz + c$, use quadratic formula:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

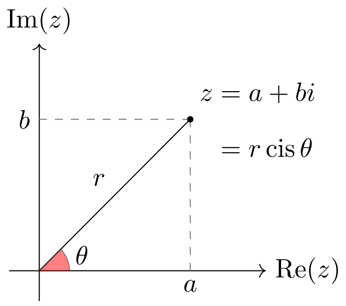
Fundamental theorem of algebra

A polynomial of degree n can be factorised into n linear factors in $\mathbb{C}$:

$$\implies P(z) = a_n(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)\ldots(z - \alpha_n)$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \in \mathbb{C}$

Argand planes



- Multiplication by $i \implies$ CCW rotation of $\frac{\pi}{2}$
- Addition: $z_1 + z_2 \equiv \overrightarrow{Oz_1} + \overrightarrow{Oz_2}$

Sketching complex graphs

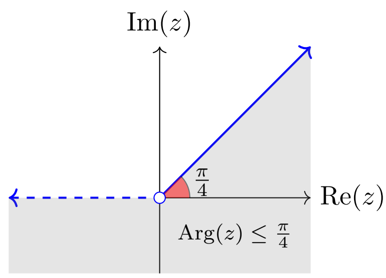
Linear

- $\text{Re}(z) = c$ or $\text{Im}(z) = c$ (perpendicular bisector)
- $\text{Im}(z) = m\,\text{Re}(z)$
- $|z + a| = |z + b| \implies 2(a - b)x = b^2 - a^2$
  Geometric: equidistant from a, b

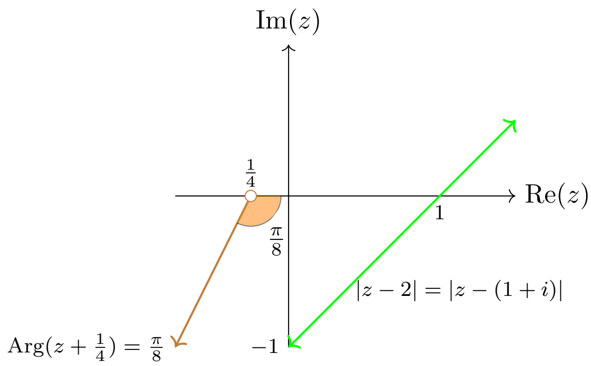
Circles

- $|z - z_1|^2 = c^2|z_2 + 2|^2$
- $|z - (a + bi)| = c \implies (x - a)^2 + (y - b)^2 = c^2$

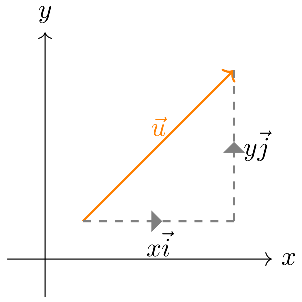
Loci $\text{Arg}(z) < \theta$



Rays $\text{Arg}(z - b) = \theta$



2 Vectors



Column notation

$$\begin{bmatrix} x \\ y \end{bmatrix} \iff x\boldsymbol{i} + y\boldsymbol{j}$$

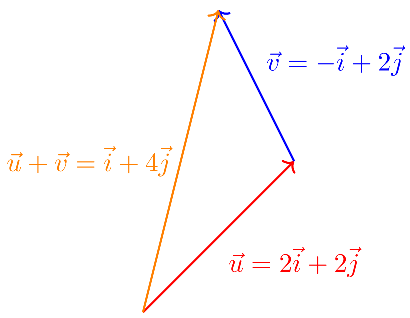
$$\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} \quad \text{between } A(x_1, y_1),\ B(x_2, y_2)$$

Scalar multiplication

$$k \cdot (x\boldsymbol{i} + y\boldsymbol{j}) = kx\boldsymbol{i} + ky\boldsymbol{j}$$

For $k \in \mathbb{R}^-$, direction is reversed

Vector addition



$$(x\boldsymbol{i} + y\boldsymbol{j}) \pm (a\boldsymbol{i} + b\boldsymbol{j}) = (x \pm a)\boldsymbol{i} + (y \pm b)\boldsymbol{j}$$

- Draw each vector head to tail then join lines
- Addition is commutative (parallelogram)
- $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v}) \implies \overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

Magnitude

$$|(x\boldsymbol{i} + y\boldsymbol{j})| = \sqrt{x^2 + y^2}$$

Parallel vectors

$$\boldsymbol{u} || \boldsymbol{v} \iff \boldsymbol{u} = k\boldsymbol{v} \text{ where } k \in \mathbb{R} \setminus \{0\}$$

For parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{cases} |\boldsymbol{a}||\boldsymbol{b}| & \text{if same direction} \\ -|\boldsymbol{a}||\boldsymbol{b}| & \text{if opposite directions} \end{cases}$$

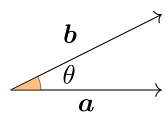
Perpendicular vectors

$$\boldsymbol{a} \perp \boldsymbol{b} \iff \boldsymbol{a} \cdot \boldsymbol{b} = 0 \quad (\text{since } \cos 90 = 0)$$

Unit vector $|\hat{\boldsymbol{a}}| = 1$

$$\begin{aligned} \hat{\boldsymbol{a}} &= \frac{1}{|\boldsymbol{a}|}\boldsymbol{a} \\ &= \boldsymbol{a} \cdot |\boldsymbol{a}| \end{aligned}$$

Scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$



$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= a_1 b_1 + a_2 b_2 \\ &= |\boldsymbol{a}||\boldsymbol{b}| \cos\theta \\ &\quad (0 \le \theta \le \pi) \text{ - from cosine rule} \end{aligned}$$

On CAS: `dotP([a b c], [d e f])`

Properties

1. $k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (k\boldsymbol{b})$
2. $\boldsymbol{a} \cdot \boldsymbol{0} = 0$
3. $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
4. $\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1$
5. $\boldsymbol{a} \cdot \boldsymbol{b} = 0 \quad \implies \quad \boldsymbol{a} \perp \boldsymbol{b}$
6. $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2 = a^2$

Angle between vectors

$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{a_1 b_1 + a_2 b_2}{|\boldsymbol{a}||\boldsymbol{b}|}$$

On CAS: `angle([a b c], [a b c])`

(Action → Vector →Angle)

Angle between vector and axis

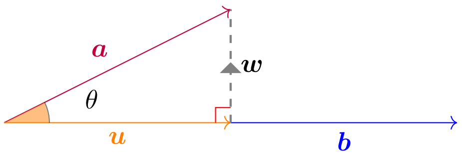
For $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ which makes angles α, β, γ with positive side of x, y, z axes:

$$\cos\alpha = \frac{a_1}{|\boldsymbol{a}|}, \quad \cos\beta = \frac{a_2}{|\boldsymbol{a}|}, \quad \cos\gamma = \frac{a_3}{|\boldsymbol{a}|}$$

On CAS: `angle([a b c], [1 0 0])`

for angle between $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ and x-axis

Projections & resolutes



$\parallel \boldsymbol{b}$ (vector projection/resolute)

$$\begin{aligned}\boldsymbol{u} &= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|^2}\boldsymbol{b} \\ &= \left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|}\right)\left(\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right) \\ &= (\boldsymbol{a}\cdot\hat{\boldsymbol{b}})\hat{\boldsymbol{b}}\end{aligned}$$

$\perp \boldsymbol{b}$ (perpendicular projection)

$$\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u}$$

$|\boldsymbol{u}|$ (scalar projection/resolute)

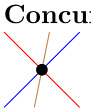
$$\begin{aligned}s &= |\boldsymbol{u}| \\ &= \boldsymbol{a}\cdot\hat{\boldsymbol{b}} \\ &= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|} \\ &= |\boldsymbol{a}|\cos\theta\end{aligned}$$

Rectangular $(\parallel, \perp)$ components

$$\boldsymbol{a} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}\cdot\boldsymbol{b}}\boldsymbol{b} + \left(\boldsymbol{a} - \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}\cdot\boldsymbol{b}}\boldsymbol{b}\right)$$

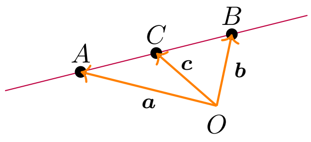
Vector proofs

Concurrent: intersection of ≥ 3 lines



Collinear points

≥ 3 points lie on the same line



e.g. Prove that

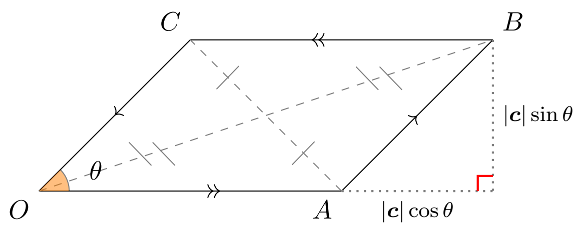
$$\begin{aligned}\overrightarrow{AC} = m\overrightarrow{AB} &\iff \boldsymbol{c} = (1-m)\boldsymbol{a} + m\boldsymbol{b} \\ &\implies \boldsymbol{c} = \overrightarrow{OA} + \overrightarrow{AC} \\ &\qquad = \overrightarrow{OA} + m\overrightarrow{AB} \\ &\qquad = \boldsymbol{a} + m(\boldsymbol{b} - \boldsymbol{a}) \\ &\qquad = \boldsymbol{a} + m\boldsymbol{b} - m\boldsymbol{a} \\ &\qquad = (1-m)\boldsymbol{a} + m\boldsymbol{b}\end{aligned}$$

$$\text{Also,} \quad \implies \overrightarrow{OC} = \lambda\vec{OA} + \mu\overrightarrow{OB}$$

$$\text{where } \lambda + \mu = 1$$

$$\text{If } C \text{ lies along } \overrightarrow{AB}, \implies 0 < \mu < 1$$

Parallelograms



- Diagonals $\overrightarrow{OB}, \overrightarrow{AC}$ bisect each other
- If diagonals are equal length, it is a rectangle
- $|\overrightarrow{OB}|^2 + |\overrightarrow{CA}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{AB}|^2 + |\overrightarrow{CB}|^2 + |\overrightarrow{OC}|^2$
- Area $= \boldsymbol{c}\cdot\boldsymbol{a}$

Useful vector properties

- $\boldsymbol{a} \parallel \boldsymbol{b} \implies \boldsymbol{b} = k\boldsymbol{a}$ for some $k \in \mathbb{R}\setminus\{0\}$
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel with at least one point in common, then they lie on the same straight line
- $\boldsymbol{a} \perp \boldsymbol{b} \iff \boldsymbol{a}\cdot\boldsymbol{b} = 0$
- $\boldsymbol{a}\cdot\boldsymbol{a} = |\boldsymbol{a}|^2$

Linear dependence

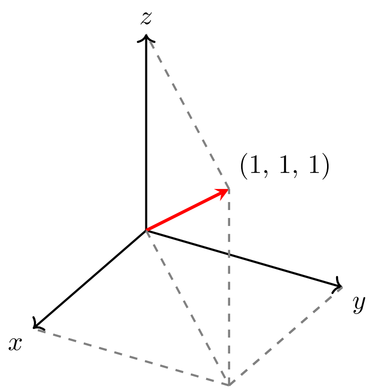
$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are linearly dependent if they are $\nparallel$ and:

$$\begin{aligned} 0 &= k\boldsymbol{a} + l\boldsymbol{b} + m\boldsymbol{c} \\ \therefore \boldsymbol{c} &= m\boldsymbol{a} + n\boldsymbol{b} \quad \text{(simultaneous)} \end{aligned}$$

$\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$ are linearly independent if no vector in the set is expressible as a linear combination of other vectors in set, or if they are parallel.

Three-dimensional vectors

Right-hand rule for axes: z is up or out of page.



Parametric vectors

Parametric equation of line through point (x_0, y_0, z_0) and parallel to $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ is:

$$\begin{cases} x = x_o + a \cdot t \\ y = y_0 + b \cdot t \\ z = z_0 + c \cdot t \end{cases}$$

3 Circular functions

$\sin(bx)$ or $\cos(bx)$: period $= \frac{2\pi}{b}$

$\tan(nx)$: period $= \frac{\pi}{n}$

asymptotes at $x = \frac{(2k+1)\pi}{2n} \mid k \in \mathbb{Z}$

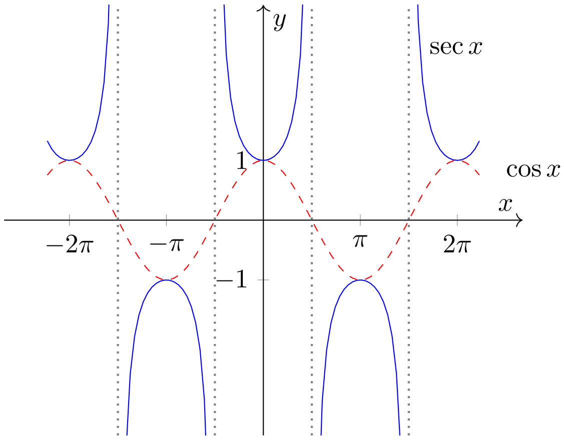
Reciprocal functions

Cosecant

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \mid \sin\theta \neq 0$$

- **Domain** $= \mathbb{R} \setminus n\pi : n \in \mathbb{Z}$
- **Range** $= \mathbb{R} \setminus (-1, 1)$
- **Turning points** at $\theta = \frac{(2n+1)\pi}{2} \mid n \in \mathbb{Z}$
- **Asymptotes** at $\theta = n\pi \mid n \in \mathbb{Z}$

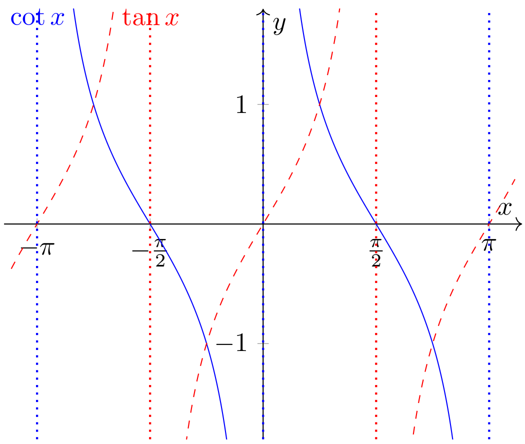
Secant



$$\sec\theta = \frac{1}{\cos\theta} \mid \cos\theta \neq 0$$

- **Domain** $= \mathbb{R} \setminus \frac{(2n+1)\pi}{2} : n \in \mathbb{Z}\}$
- **Range** $= \mathbb{R} \setminus (-1, 1)$
- **Turning points** at $\theta = n\pi \mid n \in \mathbb{Z}$
- **Asymptotes** at $\theta = \frac{(2n+1)\pi}{2} \mid n \in \mathbb{Z}$

Cotangent



$$\cot\theta = \frac{\cos\theta}{\sin\theta} \mid \sin\theta \neq 0$$

- **Domain** $= \mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$
- **Range** $= \mathbb{R}$
- **Asymptotes** at $\theta = n\pi \mid n \in \mathbb{Z}$

Symmetry properties

$$\sec(\pi \pm x) = -\sec x$$
$$\sec(-x) = \sec x$$
$$\operatorname{cosec}(\pi \pm x) = \mp \operatorname{cosec} x$$
$$\operatorname{cosec}(-x) = -\operatorname{cosec} x$$
$$\cot(\pi \pm x) = \pm \cot x$$
$$\cot(-x) = -\cot x$$

Complementary properties

$$\sec\left(\frac{\pi}{2} - x\right) = \operatorname{cosec} x$$
$$\operatorname{cosec}\left(\frac{\pi}{2} - x\right) = \sec x$$
$$\cot\left(\frac{\pi}{2} - x\right) = \tan x$$
$$\tan\left(\frac{\pi}{2} - x\right) = \cot x$$

Pythagorean identities

$$1 + \cot^2 x = \operatorname{cosec}^2 x, \quad \text{where } \sin x \neq 0$$
$$1 + \tan^2 x = \sec^2 x, \quad \text{where } \cos x \neq 0$$

Compound angle formulas

$$\cos(x \pm y) = \cos x + \cos y \mp \sin x \sin y$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

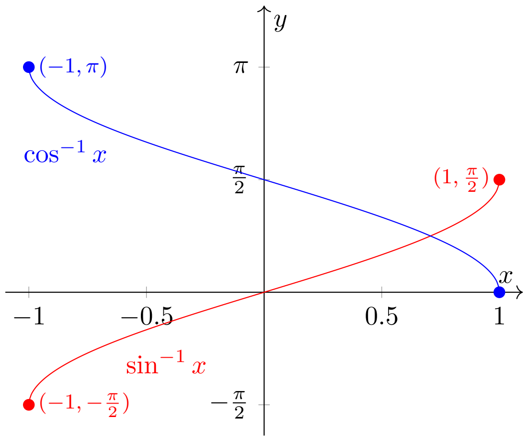
Double angle formulas

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \\ &= 2\cos^2 x - 1 \end{aligned}$$

$$\sin 2x = 2 \sin x \cos x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Inverse circular functions



Inverse functions: $f(f^{-1}(x)) = x$ (restrict domain)

$$\sin^{-1} : [-1, 1] \to \mathbb{R}, \quad \sin^{-1} x = y$$

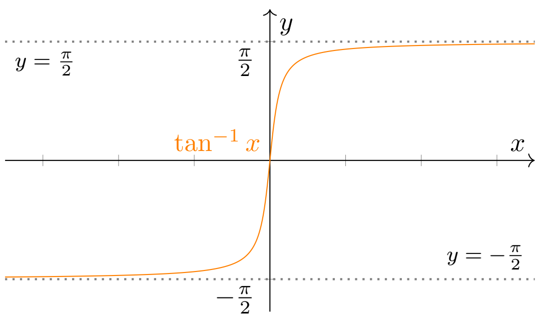
where $\sin y = x$, $y \in [\frac{-\pi}{2}, \frac{\pi}{2}]$

$$\cos^{-1} : [-1, 1] \to \mathbb{R}, \quad \cos^{-1} x = y$$

where $\cos y = x$, $y \in [0, \pi]$

$$\tan^{-1} : \mathbb{R} \to \mathbb{R}, \quad \tan^{-1} x = y$$

where $\tan y = x$, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



4 Differential calculus

Limits

$$\lim_{x \to a} f(x)$$

L^-, L^+ limit from below/above

$\lim_{x \to a} f(x)$ limit of a point

For solving $x \to \infty$, put all x terms in denominators

e.g.

$$\lim_{x \to \infty} \frac{2x+3}{x-2} = \frac{2+\frac{3}{x}}{1-\frac{2}{x}} = \frac{2}{1} = 2$$

Limit theorems

1. For constant function $f(x) = k$, $\lim_{x \to a} f(x) = k$
2. $\lim_{x \to a}(f(x) \pm g(x)) = F \pm G$
3. $\lim_{x \to a}(f(x) \times g(x)) = F \times G$
4. $\therefore \lim_{x \to a} c \times f(x) = cF$ where $c =$ constant
5. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G}, G \neq 0$
6. $f(x)$ is continuous $\iff L^- = L^+ = f(x) \, \forall x$

Gradients of secants and tangents

Secant (chord) - line joining two points on curve

Tangent - line that intersects curve at one point

First principles derivative

$$f'(x) = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

Logarithmic identities

$\log_b(xy) = \log_b x + \log_b y$

$\log_b x^n = n \log_b x$

$\log_b y^{x^n} = x^n \log_b y$

Index identities

$b^{m+n} = b^m \cdot b^n$

$(b^m)^n = b^{m \cdot n}$

$(b \cdot c)^n = b^n \cdot c^n$

$a^m \div a^n = a^{m-n}$

Reciprocal derivatives

$$\frac{1}{\frac{dy}{dx}} = \frac{dx}{dy}$$

Differentiating $x = f(y)$

Find $\frac{dx}{dy}$, then $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$

Second derivative

$$f(x) \longrightarrow f'(x) \longrightarrow f''(x)$$
$$\implies y \longrightarrow \frac{dy}{dx} \longrightarrow \frac{d^2y}{dx^2}$$

Order of polynomial nth derivative decrements each time the derivative is taken

Points of Inflection

Stationary point - i.e. $f'(x) = 0$

Point of inflection - max |gradient| (i.e. $f'' = 0$)

Strictly increasing/decreasing

For x_2 and x_1 where $x_2 > x_1$:

strictly increasing

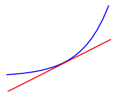
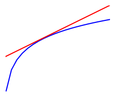
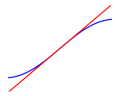
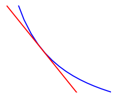
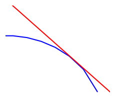
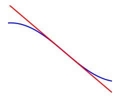
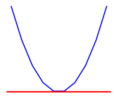
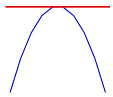
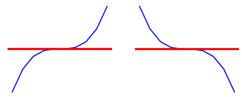
where $f(x_2) > f(x_1)$ or $f'(x) > 0$

strictly decreasing

where $f(x_2) < f(x_1)$ or $f'(x) < 0$

> **Endpoints are included, even where $\frac{dy}{dx} = 0$**

- $f'(a) = 0, \ f''(a) > 0$
  local min at $(a, f(a))$ (concave up)
- $f'(a) = 0, \ f''(a) < 0$
  local max at $(a, f(a))$ (concave down)
- $f''(a) = 0$
  point of inflection at $(a, f(a))$
- $f''(a) = 0, \ f'(a) = 0$
  stationary point of inflection at $(a, f(a)$

	$\frac{d^2y}{dx^2} > 0$	$\frac{d^2y}{dx^2} < 0$	$\frac{d^2y}{dx^2} = 0$ (inflection)
$\frac{dy}{dx} > 0$	 Rising (concave up)	 Rising (concave down)	 Rising inflection point
$\frac{dy}{dx} < 0$	 Falling (concave up)	 Falling (concave down)	 Falling inflection point
$\frac{dy}{dx} = 0$	 Local minimum	 Local maximum	 Stationary inflection point

Implicit Differentiation

Used for differentiating circles etc.

If p and q are expressions in x and y such that $p = q$, for all x and y, then:

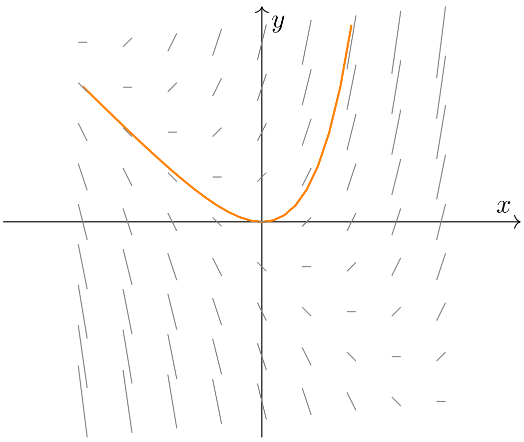
$$\frac{dp}{dx} = \frac{dq}{dx} \quad \text{and} \quad \frac{dp}{dy} = \frac{dq}{dy}$$

On CAS

Action → Calculation

```
impDiff(y^2+ax=5, x, y)
```

Slope fields



Parametric equations

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \text{ provided } \frac{dx}{dt} \neq 0$$

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{dy'}{dt}\right)}{\left(\frac{dx}{dt}\right)} \text{ where } y' = \frac{dy}{dx}$$

Integration

$$\int f(x) \cdot dx = F(x) + c \quad \text{where } F'(x) = f(x)$$

Definite integrals

$$\int_a^b f(x) \cdot dx = [F(x)]_a^b = F(b) - F(a)$$

- Signed area enclosed by
  $y = f(x), \quad y = 0, \quad x = a, \quad x = b.$

- *Integrand* is f.

Properties

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$
$$\int_a^a f(x)\,dx = 0$$
$$\int_a^b k\cdot f(x)\,dx = k\int_a^b f(x)\,dx$$
$$\int_a^b f(x)\pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Integration by substitution

$$\int f(u)\frac{du}{dx}\cdot dx = \int f(u)\cdot du$$

$f(u)$ must be 1:1 $\implies$ one x for each y

e.g. for $y = \int (2x+1)\sqrt{x+4}\cdot dx$

let $u = x+4$

$\implies \frac{du}{dx} = 1$

$\implies x = u-4$

then $y = \int (2(u-4)+1)u^{\frac{1}{2}}\cdot du$

(solve as normal integral)

Definite integrals by substitution

For $\int_a^b f(x)\frac{du}{dx}\cdot dx$, evaluate new a and b for $f(u)\cdot du$.

Trigonometric integration

$$\sin^m x\cos^n x\cdot dx$$

m is odd: $m = 2k+1$ where $k\in\mathbb{Z}$

$\implies \sin^{2k+1}x = (\sin^2 z)^k\sin x = (1-\cos^2 x)^k\sin x$

Substitute $u = \cos x$

n is odd: $n = 2k+1$ where $k\in\mathbb{Z}$

$\implies \cos^{2k+1}x = (\cos^2 x)^k\cos x = (1-\sin^2 x)^k\cos x$

Substitute $u = \sin x$

m and n are even: use identities...

- $\sin^2 x = \frac{1}{2}(1-\cos 2x)$
- $\cos^2 x = \frac{1}{2}(1+\cos 2x)$
- $\sin 2x = 2\sin x\cos x$

Partial fractions

To factorise $f(x) = \frac{\delta}{\alpha\cdot\beta}$:

$$\frac{\delta}{\alpha\cdot\beta\cdot\gamma} = \frac{A}{\alpha}+\frac{B}{\beta}+\frac{C}{\gamma} \tag{1}$$

Multiply by $(\alpha\cdot\beta\cdot\gamma)$:

$$\delta = \beta\gamma A + \alpha\gamma B + \alpha\beta C \tag{2}$$

Substitute $x = \{\alpha,\beta,\gamma\}$ into (2) to find denominators

Repeated linear factors

$$\frac{p(x)}{(x-a)^n} = \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}$$

Irreducible quadratic factors

$$\text{e.g.}\quad \frac{3x-4}{(2x-3)(x^2+5)} = \frac{A}{2x-3} + \frac{Bx+C}{x^2+5}$$

On CAS

Action → Transformation:

```
expand(..., x)
```

To reverse, use `combine(...)`

Graphing integrals on CAS

On CAS

In main: Interactive → Calculation → $\int$

For restrictions, `Define f(x)=...` then `f(x)|x>...`

Applications of antidifferentiation

- x-intercepts of $y = f(x)$ identify x-coordinates of stationary points on $y = F(x)$
- nature of stationary points is determined by sign of $y = f(x)$ on either side of its x-intercepts
- if $f(x)$ is a polynomial of degree n, then $F(x)$ has degree $n+1$

To find stationary points of a function, substitute x value of given point into derivative. Solve for $\frac{dy}{dx} = 0$. Integrate to find original function.

Solids of revolution

Approximate as sum of infinitesimally-thick cylinders

Rotation about x-axis

$$V = \pi \int_{x=a}^{x=b} f(x)^2 \, dx$$

Rotation about y-axis

$$\begin{aligned} V &= \pi \int_{y=a}^{y=b} x^2 \, dy \\ &= \pi \int_{y=a}^{y=b} (f(y))^2 \, dy \end{aligned}$$

Regions not bound by $y = 0$

$$V = \pi \int_a^b f(x)^2 - g(x)^2 \, dx$$

where $f(x) > g(x)$

Length of a curve

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} \, dx \quad \text{(Cartesian)}$$

$$L = \int_a^b \sqrt{\frac{dx}{dt} + (\frac{dy}{dt})^2} \, dt \quad \text{(parametric)}$$

On CAS

a) Evaluate formula

b) Interactive → Calculation → Line → `arcLen`

Rates

Gradient at a point on parametric curve

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} \mid \frac{dx}{dt} \neq 0 \text{ (chain rule)}$$

$$\frac{d^2}{dx^2} = \frac{d(y')}{dx} = \frac{dy'}{dt} \div \frac{dx}{dt} \mid y' = \frac{dy}{dx}$$

Rational functions

$$f(x) = \frac{P(x)}{Q(x)} \quad \text{where } P, Q \text{ are polynomial functions}$$

Fundamental theorem of calculus

If f is continuous on $[a, b]$, then

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

where $F = \int f \, dx$

Differential equations

Order - highest power inside derivative

Degree - highest power of highest derivative

e.g. $\left(\frac{dy^2}{d^2} x\right)^3$ order 2, degree 3

To verify solutions, find $\frac{dy}{dx}$ from y and substitute into original

Function of the dependent variable

If $\frac{dy}{dx} = g(y)$, then $\frac{dx}{dy} = 1 \div \frac{dy}{dx} = \frac{1}{g(y)}$. Integrate both sides to solve equation. Only add c on one side. Express e^c as A.

Mixing problems

$$\left(\frac{dm}{dt}\right)_\Sigma = \left(\frac{dm}{dt}\right)_{\text{in}} - \left(\frac{dm}{dt}_{\text{out}}\right)$$

Separation of variables

If $\frac{dy}{dx} = f(x)g(y)$, then:

$$\int f(x) \, dx = \int \frac{1}{g(y)} \, dy$$

Euler's method for solving DEs

$$\frac{f(x+h) - f(x)}{h} \approx f'(x) \quad \text{for small } h$$

$$\implies f(x+h) \approx f(x) + hf'(x)$$

Derivatives

$f(x)$	$f'(x)$	
$\sin x$	$\cos x$	
$\sin ax$	$a \cos ax$	
$\cos x$	$-\sin x$	
$\cos ax$	$-a \sin ax$	
$\tan f(x)$	$f^2(x) \sec^2 f(x)$	
e^x	e^x	
e^{ax}	ae^{ax}	
ax^{nx}	$an \cdot e^{nx}$	
$\log_e x$	$\frac{1}{x}$	
$\log_e ax$	$\frac{1}{x}$	
$\log_e f(x)$	$\frac{f'(x)}{f(x)}$	
$\sin(f(x))$	$f'(x) \cdot \cos(f(x))$	
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$	
$\tan^{-1} x$	$\frac{1}{1+x^2}$	
$\frac{d}{dy} f(y)$	$\frac{1}{\frac{dx}{dy}}$	(reciprocal)
uv	$u\frac{dv}{dx} + v\frac{du}{dx}$	(product rule)
$\frac{u}{v}$	$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	(quotient rule)
$f(g(x))$	$f'(g(x)) \cdot g'(x)$	

Antiderivatives

$f(x)$	$\int f(x) \cdot dx$	
k (constant)	$kx + c$	
x^n	$\frac{1}{n+1} x^{n+1}$	
ax^{-n}	$a \cdot \log_e \lvert x \rvert + c$	
$\frac{1}{ax+b}$	$\frac{1}{a} \log_e(ax+b) + c$	
$(ax+b)^n$	$\frac{1}{a(n+1)}(ax+b)^{n-1} + c \mid n \neq 1$	
$(ax+b)^{-1}$	$\frac{1}{a} \log_e \lvert ax+b \rvert + c$	
e^{kx}	$\frac{1}{k} e^{kx} + c$	
e^k	$e^k x + c$	
$\sin kx$	$\frac{-1}{k} \cos(kx) + c$	
$\cos kx$	$\frac{1}{k} \sin(kx) + c$	
$\sec^2 kx$	$\frac{1}{k} \tan(kx) + c$	
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1} \frac{x}{a} + c \mid a > 0$	
$\frac{-1}{\sqrt{a^2-x^2}}$	$\cos^{-1} \frac{x}{a} + c \mid a > 0$	
$\frac{a}{a^2-x^2}$	$\tan^{-1} \frac{x}{a} + c$	
$\frac{f'(x)}{f(x)}$	$\log_e f(x) + c$	
$\int f(u) \cdot \frac{du}{dx} \cdot dx$	$\int f(u) \cdot du$	(substitution)
$f(x) \cdot g(x)$	$\int [f'(x) \cdot g(x)] dx + \int [g'(x) f(x)] dx$	

Note $\sin^{-1}\left(\frac{x}{a}\right) + \cos^{-1}\left(\frac{x}{a}\right)$ is constant $\forall\, x \in (-a, a)$

5 Kinematics & Mechanics

Constant acceleration

- **Position** - relative to origin
- **Displacement** - relative to starting point

Velocity-time graphs

Displacement: *signed* area

Distance travelled: *total* area

$$\text{acceleration} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

	no
$v = u + at$	x
$v^2 = u^2 + 2as$	t
$s = \frac{1}{2}(v+u)t$	a
$s = ut + \frac{1}{2}at^2$	v
$s = vt - \frac{1}{2}at^2$	u

$$v_{\text{avg}} = \frac{\Delta\text{position}}{\Delta t}$$

$$\begin{aligned}\text{speed} &= |\text{velocity}| \\ &= \sqrt{v_x^2 + v_y^2 + v_z^2}\end{aligned}$$

Distance travelled between $t = a \to t = b$:

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$$

Shortest distance between $\boldsymbol{r}(t_0)$ and $\boldsymbol{r}(t_1)$:

$$= |\boldsymbol{r}(t_1) - \boldsymbol{r}(t_2)|$$

Vector functions

$$\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

- If $\boldsymbol{r}(t) \equiv$ position with time, then the graph of endpoints of $\boldsymbol{r}(t) \equiv$ Cartesian path
- Domain of $\boldsymbol{r}(t)$ is the range of $x(t)$
- Range of $\boldsymbol{r}(t)$ is the range of $y(t)$

Vector calculus

Derivative

Let $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{(}j)$. If both $x(t)$ and $y(t)$ are differentiable, then:

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$$

6 Dynamics

Resolution of forces

Resultant force is sum of force vectors

In angle-magnitude form

$$\begin{array}{ll}\text{Cosine rule:} & c^2 = a^2 + b^2 - 2ab\cos\theta \\ \text{Sine rule:} & \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}\end{array}$$

In $\boldsymbol{i}$—$\boldsymbol{j}$ form

Vector of a N at θ to x axis is equal to $a\cos\theta\boldsymbol{i} + a\sin\theta\boldsymbol{j}$.

Convert all force vectors then add.

To find angle of an $a\boldsymbol{i} + b\boldsymbol{j}$ vector, use $\theta = \tan^{-1}\frac{b}{a}$

Resolving in a given direction

The resolved part of a force P at angle θ is has magnitude $P\cos\theta$

To convert force $||\vec{OA}$ to angle-magnitude form, find component $\perp \vec{OA}$ then:

$$|\boldsymbol{r}| = \sqrt{\left(||\vec{OA}\right)^2 + \left(\perp \vec{OA}\right)^2}$$

$$\theta = \tan^{-1}\frac{\perp \vec{OA}}{||\vec{OA}}$$

Newton's laws

1. Velocity is constant without ΣF
2. $\frac{d}{dt}\rho \propto \Sigma F \implies \boldsymbol{F} = m\boldsymbol{a}$
3. Equal and opposite forces

Weight

A mass of m kg has force of mg acting on it

Momentum ρ

$$\rho = mv \qquad \text{(units kg m/s or Ns)}$$

Reaction force R

- With no vertical velocity, $R = mg$
- With vertical acceleration, $|R| = m|a| - mg$
- With force F at angle θ, then $R = mg - F\sin\theta$

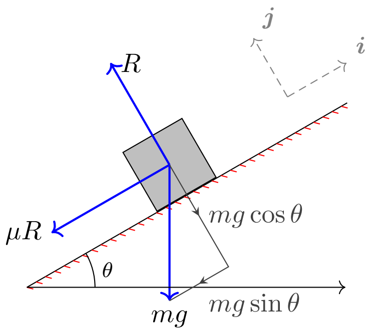
Friction

$$F_R = \mu R \qquad \text{(friction coefficient)}$$

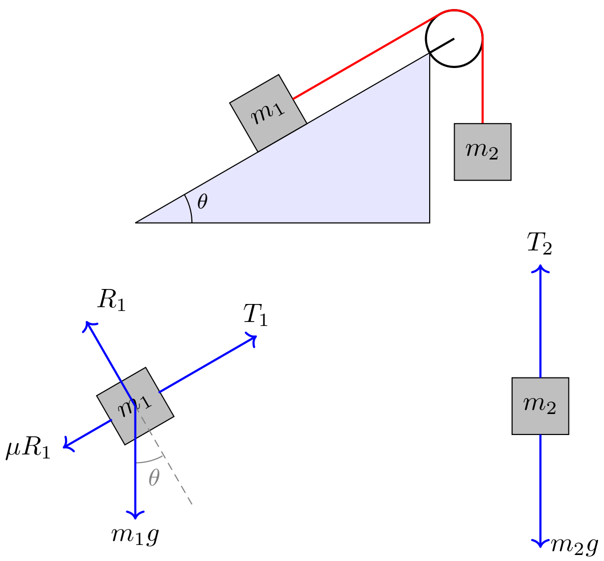
Inclined planes

$$\boldsymbol{F} = |\boldsymbol{F}|\cos\theta\boldsymbol{i} + |\boldsymbol{F}|\sin\theta\boldsymbol{j}$$

- Normal force R is at right angles to plane
- Let direction up the plane be $\boldsymbol{i}$ and perpendicular to plane $\boldsymbol{j}$



Connected particles



- **Suspended pulley:** tension in both sections of rope are equal
  $|a| = g\dfrac{m_1 - m_2}{m_1 + m_2}$ where m_1 accelerates down

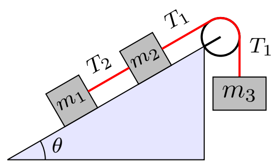
$$\left\{\begin{aligned} m_1g - T &= m_1a \\ T - m_2g &= m_2a \end{aligned}\right\} \implies m_1g - m_2g = m_1a + m_2a$$

- **String pulling mass on inclined pane:** Resolve parallel to plane

$$T - mg\sin\theta = ma$$

- **Linear connection:** find acceleration of system first
- **Pulley on right angle:** $a = \frac{m_2g}{m_1+m_2}$ where m_2 is suspended (frictionless on both surfaces)
- **Pulley on edge of incline:** find downwards force W_2 and components of mass on plane

In this example, note $T_1 \neq T_2$:



Equilibrium

$$\frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c} \qquad \text{(Lami's theorem)}$$

$$c^2 = a^2 + b^2 - 2ab\cos\theta \qquad \text{(cosine rule)}$$

Three methods:

1. Lami's theorem (sine rule)
2. Triangle of forces (cosine rule)
3. Resolution of forces ($\Sigma F = 0$ - simultaneous)

On CAS

To verify: Geometry tab, then select points with normal cursor. Click right arrow at end of toolbar and input point, then lock known constants.

Variable forces (DEs)

$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

7 Statistics

Continuous random variables

A continuous random variable X has a pdf f such that:

1. $f(x) \geq 0 \forall x$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$E(X) = \int_{\mathbf{X}} (x \cdot f(x))\,dx$$
$$\mathrm{Var}(X) = E\left[(X-\mu)^2\right]$$

$$\Pr(X \leq c) = \int_{-\infty}^{c} f(x)\,dx$$

Two random variables X, Y

If X and Y are independent:

$$\mathrm{E}(aX + bY) = a\,\mathrm{E}(X) + b\,\mathrm{E}(Y)$$
$$\mathrm{Var}(aX \pm bY \pm c) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y)$$

Linear functions $X \to aX + b$

$$\begin{aligned}\Pr(Y \leq y) &= \Pr(aX + b \leq y)\\ &= \Pr\left(X \leq \frac{y-b}{a}\right)\\ &= \int_{-\infty}^{\frac{y-b}{a}} f(x)\,dx\end{aligned}$$

Mean: $\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b$

Variance: $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$

Expectation theorems

For some non-linear function g, the expected value $E(g(X))$ is not equal to $g(E(X))$.

$$\begin{aligned}E(X^2) &= \mathrm{Var}(X) - [E(X)]^2 &\\ E(X^n) &= \Sigma x^n \cdot p(x) & \text{(non-linear)}\\ &\neq [E(X)]^n &\\ E(aX \pm b) &= aE(X) \pm b & \text{(linear)}\\ E(b) &= b & (\forall b \in \mathbb{R})\\ E(X+Y) &= E(X) + E(Y) & \text{(two variables)}\end{aligned}$$

Sample mean

Approximation of the **population mean** determined experimentally.

$$\overline{x} = \frac{\Sigma x}{n}$$

where

n is the size of the sample (number of sample points)

x is the value of a sample point

On CAS

1. Spreadsheet
2. In cell A1:
   `mean(randNorm(sd, mean, sample size))`
3. Edit → Fill → Fill Range
4. Input range as A1:An where n is the number of samples
5. Graph → Histogram

Sample size of n

$$\overline{X} = \sum_{i=1}^{n} \frac{x_i}{n} = \frac{\sum x}{n}$$

Sample mean is distributed with mean μ and sd $\frac{\sigma}{\sqrt{n}}$ (approaches these values for increasing sample size n).

For a new distribution with mean of n trials, $\mathrm{E}(X') = \mathrm{E}(X)$, $\quad \mathrm{sd}(X') = \dfrac{\mathrm{sd}(X)}{\sqrt{n}}$

On CAS

- Spreadsheet → Catalog → `randNorm(sd, mean, n)` where `n` is the number of samples. Show histogram with Histogram key in top left
- To calculate parameters of a dataset: Calc → One-variable

Normal distributions

$$Z = \frac{X - \mu}{\sigma}$$

Normal distributions must have area (total prob.) of 1 $\implies \int_{-\infty}^{\infty} f(x)\,dx = 1$

mean = mode = median

Always express z as +ve. Express confidence interval as ordered pair.

Central limit theorem

If X is randomly distributed with mean μ and sd σ, then with an adequate sample size n the distribution of the sample mean $\overline{X}$ is approximately normal with mean $E(\overline{X})$ and $\text{sd}(\overline{X}) = \frac{\sigma}{\sqrt{n}}$.

Confidence intervals

- **Point estimate:** single-valued estimate of the population mean from the value of the sample mean $\overline{x}$
- **Interval estimate:** confidence interval for population mean μ
- $C\%$ confidence interval $\implies$ $C\%$ of samples will contain population mean μ

95% confidence interval

For 95% c.i. of population mean μ:

$$x \in \left(\overline{x} \pm 1.96\frac{\sigma}{\sqrt{n}}\right)$$

where:

$\overline{x}$ is the sample mean

σ is the population sd

n is the sample size from which $\overline{x}$ was calculated

On CAS

Menu → Stats → Calc → Interval

Set *Type = One-Sample Z Int*

and select *Variable*

Margin of error

For 95% confidence interval of μ:

$$M = 1.96 \times \frac{\sigma}{\sqrt{n}}$$

$$\implies n = \left(\frac{1.96\sigma}{M}\right)^2$$

Always round n up to a whole number of samples.

General case

For $C\%$ c.i. of population mean μ:

$$x \in \left(\overline{x} \pm k\frac{\sigma}{\sqrt{n}}\right)$$

where k is such that $\Pr(-k < Z < k) = \frac{C}{100}$

Confidence interval for multiple trials

For a set of n confidence intervals (samples), there is 0.95^n chance that all n intervals contain the population mean μ.

8 Hypothesis testing

Note hypotheses are always expressed in terms of population parameters

Null hypothesis H_0

Sample drawn from population has same mean as control population, and any difference can be explained by sample variations.

Alternative hypothesis H_1

Amount of variation from control is significant, despite standard sample variations.

p-value

Probability of observing a value of the sample statistic as significant as the one observed, assuming null hypothesis is true.

For one-tail tests:

$$\begin{aligned} p\text{-value} &= \Pr\left(\overline{X} \lesseqgtr \mu(\mathbf{H}_1) \mid \mu = \mu(\mathbf{H}_0)\right) \\ &= \Pr\left(Z \lesseqgtr \frac{(\mu(\mathbf{H}_1) - \mu(\mathbf{H}_0)) \cdot \sqrt{n}}{\operatorname{sd}(X)}\right) \end{aligned}$$

then use `normCdf` with std. norm.

p	Conclusion
> 0.05	insufficient evidence against $\mathbf{H}_0$
< 0.05 (5%)	good evidence against $\mathbf{H}_0$
< 0.01 (1%)	strong evidence against $\mathbf{H}_0$
< 0.001 (0.1%)	very strong evidence against $\mathbf{H}_0$

Significance level α

The condition for rejecting the null hypothesis.

If $p < \alpha$, null hypothesis is **rejected**

If $p > \alpha$, null hypothesis is **accepted**

z-test

Hypothesis test for a mean of a sample drawn from a normally distributed population with a known standard deviation.

On CAS

Menu → Statistics → Calc → Test.

Select *One-Sample Z-Test* and *Variable*, then input:

μ cond:	same operator as $\mathbf{H}_1$
μ_0:	expected sample mean (null hypothesis)
σ:	standard deviation (null hypothesis)
$\overline{x}$:	sample mean
n:	sample size

One-tail and two-tail tests

$$p\text{-value (two-tail)} = 2 \times p\text{-value (one-tail)}$$

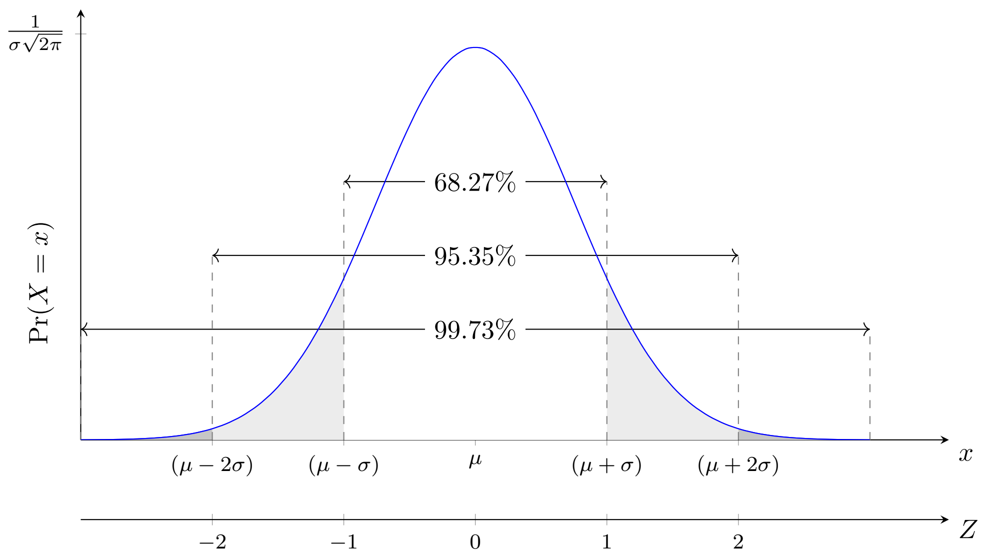
One tail

- μ has changed in one direction
- State "$\mathbf{H}_1 : \mu \lesseqgtr$ known population mean"

Two tail

- Direction of $\Delta\mu$ is ambiguous
- State "$\mathbf{H}_1 : \mu \neq$ known population mean"

$$\begin{aligned} p\text{-value} &= \Pr(|\overline{X} - \mu| \geq |\overline{x}_0 - \mu|) \\ &= \left(|Z| \geq \left|\frac{\overline{x}_0 - \mu}{\sigma \div \sqrt{n}}\right|\right) \end{aligned}$$



where

μ is the population mean under $\mathbf{H}_0$

$\overline{x}_0$ is the observed sample mean

σ is the population s.d.

n is the sample size

Modulus notation for two tail

$\Pr(|\overline{X} - \mu| \geq a) \implies$ "the probability that the distance between $\overline{\mu}$ and μ is $\geq a$"

Inverse normal

On CAS

invNormCdf("L", α, $\dfrac{\sigma}{n^\alpha}$, μ)

Errors

Type I error $\mathbf{H}_0$ is rejected when it is **true**

Type II error $\mathbf{H}_0$ is **not** rejected when it is **false**

	Actual result	
***z*-test**	$\mathbf{H}_0$ true	$\mathbf{H}_0$ false
Reject $\mathbf{H}_0$	Type I error	Correct
Do not reject $\mathbf{H}_0$	Correct	Type II error