Complex & Imaginary Numbers

Imaginary numbers

$$i^2 = -1 \quad \therefore i = \sqrt{-1}$$

Simplifying negative surds

$$\begin{aligned} \sqrt{-2} &= \sqrt{-1 \times 2} \\ &= \sqrt{2}i \end{aligned} \tag{1}$$

Complex numbers

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

General form: $z = a + bi$
$\text{Re}(z) = a, \quad \text{Im}(z) = b$

Addition

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 + z_2 = (a + c) + (b + d)i$$

Subtraction

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 - z_2 = (a - c) + (b - d)i$$

Multiplication by a real constant

If $z = a + bi$ and $k \in \mathbb{R}$, then

$$kz = ka + kbi$$

Powers of i

- $i^{4n} = 1$
- $i^{4n+1} = i$
- $i^{4n+2} = -1$
- $i^{4n+3} = -i$

For i^n, find remainder r when $n \div 4$. Then $i^n = i^r$.

Multiplying complex expressions

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 \times z_2 = (ac - bd) + (ad + bc)i$$

Conjugates

$$\overline{z} = a \mp bi$$

Properties

- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$
- $\overline{kz} = k\overline{z}$, for $k \in \mathbb{R}$
- $z\overline{z} == (a + bi)(a - bi) = a^2 + b^2 = |z|^2$
- $z + \overline{z} = 2\,\text{Re}(z)$

Modulus

Distance from origin.

$$|z| = \sqrt{a^2 + b^2} \quad \therefore z\overline{z} = |z|^2$$

Properties

- $|z_1 z_2| = |z_1||z_2|$
- $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}$
- $|z_1 + z_2| \leq |z_1 + |z_2|$

Multiplicative inverse

$$\begin{aligned} z^{-1} &= \frac{1}{z} \\ &= \frac{a - bi}{a^2 + B^2} \\ &= \frac{\overline{z}}{|z|^2} \end{aligned} \tag{2}$$

Dividing complex numbers

$$\frac{z_1}{z_2} = z_1 \ {z_2}^{-1} = \frac{z_1 \overline{z_2}}{|z_2|^2} \quad \text{(multiplicative inverse)}$$

In practice, rationalise denominator:

$$\frac{z_1}{z_2} = \frac{(a + bi)(c - di)}{c^2 + d^2}$$

Argand planes

- Geometric representation of $\mathbb{C}$
- horizontal = $\text{Re}(z)$; vertical = $\text{Im}(z)$
- Multiplication by i results in an anticlockwise rotation of $\frac{\pi}{2}$

Complex polynomials

Include $\pm$ for all solutions, including imaginary

Sum of two squares (quadratics)

$$z^2 + a^2 = z^2 - (ai)^2 = (z + ai)(z - ai)$$

Complete the square to get to this point.

Dividing complex polynomials

$P(z) \div D(z)$ gives quotient $Q(z)$ and remainder $R(z)$:

$$P(z) = D(z)Q(z) + R(z)$$

Remainder theorem

Let $\alpha \in \mathbb{C}$. Remainder of $P(z) \div (z - \alpha)$ is $P(\alpha)$

Factor theorem

If $a + bi$ is a solution to $P(z) = 0$, then:

- $P(a + bi) = 0$
- $z - (a + bi)$ is a factor of $P(z)$

Sum of two cubes

$$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$$

Conjugate root theorem

If $a + bi$ is a solution to $P(z) = 0$, then the conjugate $\overline{z} = a - bi$ is also a solution.

Polar form

$$\begin{aligned} z &= r \operatorname{cis} \theta \\ &= r(\cos\theta + i \sin\theta) \\ &= a + bi \end{aligned} \tag{3}$$

- $r = |z| = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}$
- $\theta = \arg(z)$ (on CAS: `arg(a+bi)`)
- **principal argument** is $\operatorname{Arg}(z) \in (-\pi, \pi]$ (note capital Arg)

Each complex number has multiple polar representations: $z = r \operatorname{cis}\theta = r \operatorname{cis}(\theta + 2n\pi)$ with $n \in \mathbb{Z}$ revolutions

Conjugate in polar form

$$(r \operatorname{cis}\theta)^{-1} = r \operatorname{cis}(-\theta)$$

Reflection of z across horizontal axis.

Multiplication and division in polar form

$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

de Moivres' Theorem

$$(r \operatorname{cis}\theta)^n = r^n \operatorname{cis}(n\theta) \text{ where } n \in \mathbb{Z}$$

Roots of complex numbers

nth roots of $z = r \operatorname{cis}\theta$ are

$$z = r^{\frac{1}{n}} \operatorname{cis}(\frac{\theta + 2k\pi}{n})$$

Same modulus for all solutions. Arguments are separated by $\frac{2\pi}{n}$

The solutions of $z^n = a$ where $a \in \mathbb{C}$ lie on circle

$$x^2 + y^2 = (|a|^{\frac{1}{n}})^2$$

Sketching complex graphs

Straight line

- $\operatorname{Re}(z) = c$ or $\operatorname{Im}(z) = c$ (perpendicular bisector)
- $\operatorname{Arg}(z) = \theta$
- $|z + a| = |z + bi|$ where $m = \frac{a}{b}$
- $|z + a| = |z + b| \longrightarrow 2(a - b)x = b^2 - a^2$

Circle

$|z - z_1|^2 = c^2 |z_2 + 2|^2$ or $|z - (a + bi)| = c$

Locus

$\operatorname{Arg}(z) < \theta$