

Year 12 Specialist

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1 Complex numbers

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

Cartesian form: $a + bi$ Polar form: $r \operatorname{cis} \theta$

Properties

$$|z_1 z_2| = |z_1| |z_2|$$

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$$

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Operations

	Cartesian	Polar
$z_1 \pm z_2$	$(a \pm c)(b \pm d)i$	convert to $a + bi$
$+k \times z$	$ka \pm kbi$	$kr \operatorname{cis} \theta$
$-k \times z$		$kr \operatorname{cis}(\theta \pm \pi)$
$z_1 \cdot z_2$	$ac - bd + (ad + bc)i$	$r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$
$z_1 \div z_2$	$(z_1 \bar{z}_2) \div z_2 ^2$	$\left(\frac{r_1}{r_2}\right) \operatorname{cis}(\theta_1 - \theta_2)$

Multiplicative inverse

$$\begin{aligned} z^{-1} &= \frac{a - bi}{a^2 + b^2} \\ &= \frac{\bar{z}}{|z|^2} a \\ &= r \operatorname{cis}(-\theta) \end{aligned}$$

Scalar multiplication in polar form

For $k \in \mathbb{R}^+$:

$$k(r \operatorname{cis} \theta) = kr \operatorname{cis} \theta$$

For $k \in \mathbb{R}^-$:

$$k(r \operatorname{cis} \theta) = kr \operatorname{cis} \begin{cases} \theta - \pi & |0 < \operatorname{Arg}(z) \leq \pi \\ \theta + \pi & |-\pi < \operatorname{Arg}(z) \leq 0 \end{cases}$$

Dividing over \mathbb{C}

$$\begin{aligned} \frac{z_1}{z_2} &= z_1 z_2^{-1} \\ &= \frac{z_1 \bar{z}_2}{|z_2|^2} \\ &= \frac{(a + bi)(c - di)}{c^2 + d^2} \end{aligned}$$

then rationalise denominator

Conjugate

`conj(a+bi)`

$$\begin{aligned} \bar{z} &= a \mp bi \\ &= r \operatorname{cis}(-\theta) \end{aligned}$$

Properties

$$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$$

$$\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$$

$$\overline{kz} = k\bar{z} \quad \forall k \in \mathbb{R}$$

$$\begin{aligned} z\bar{z} &= (a + bi)(a - bi) \\ &= a^2 + b^2 \\ &= |z|^2 \end{aligned}$$

Modulus

$$|z| = |\vec{Oz}| = \sqrt{a^2 + b^2}$$

Polar form

$$r \operatorname{cis} \theta = r(\cos \theta + i \sin \theta)$$

- $r = |z| = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}$
- $\theta = \arg(z)$ `arg(a+bi)`
- $\operatorname{Arg}(z) \in (-\pi, \pi)$ (**principal argument**)
- Multiple representations:
 $r \operatorname{cis} \theta = r \operatorname{cis}(\theta + 2n\pi)$ with $n \in \mathbb{Z}$ revolutions
- $\operatorname{cis} \pi = -1, \quad \operatorname{cis} 0 = 1$

On CAS

$$\text{compToTrig}(a+bi) \iff \text{cExpand}\{r \cdot \operatorname{cis} X\}$$

de Moivre's' theorem

$$(r \operatorname{cis} \theta)^n = r^n \operatorname{cis}(n\theta) \quad \text{where } n \in \mathbb{Z}$$

Complex polynomials

Include \pm for all solutions, incl. imaginary

Sum of squares	$z^2 + a^2 = z^2 - (ai)^2$ $= (z + ai)(z - ai)$
Sum of cubes	$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
Division	$P(z) = D(z)Q(z) + R(z)$
Remainder theorem	Let $\alpha \in \mathbb{C}$. Remainder of $P(z) \div (z - \alpha)$ is $P(\alpha)$
Factor theorem	$z - \alpha$ is a factor of $P(z) \iff P(\alpha) = 0$ for $\alpha \in \mathbb{C}$
Conjugate root theorem	$P(z) = 0$ at $z = a \pm bi$ (\implies both z_1 and \bar{z}_1 are solutions)

Factor theorem

If $\beta z + \alpha$ is a factor of $P(z)$, then $P(-\frac{\alpha}{\beta}) = 0$.

n th roots

n th roots of $z = r \text{ cis } \theta$ are:

$$z = r^{\frac{1}{n}} \text{ cis } \left(\frac{\theta + 2k\pi}{n} \right)$$

- Same modulus for all solutions
- Arguments separated by $\frac{2\pi}{n} \therefore$ there are n roots
- If one square root is $a + bi$, the other is $-a - bi$
- Give one implicit n th root z_1 , function is $z = z_1^n$
- Solutions of $z^n = a$ where $a \in \mathbb{C}$ lie on the circle $x^2 + y^2 = \left(|a|^{\frac{1}{n}}\right)^2$ (intervals of $\frac{2\pi}{n}$)

For $0 = az^2 + bz + c$, use quadratic formula:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

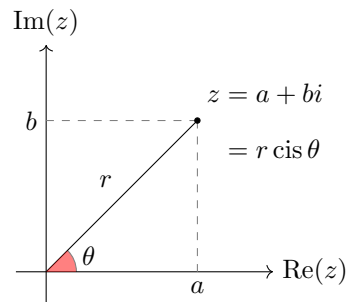
Fundamental theorem of algebra

A polynomial of degree n can be factorised into n linear factors in \mathbb{C} :

$$\implies P(z) = a_n(z - \alpha_1)(z - \alpha_2)(z - \alpha_3) \dots (z - \alpha_n)$$

where $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n \in \mathbb{C}$

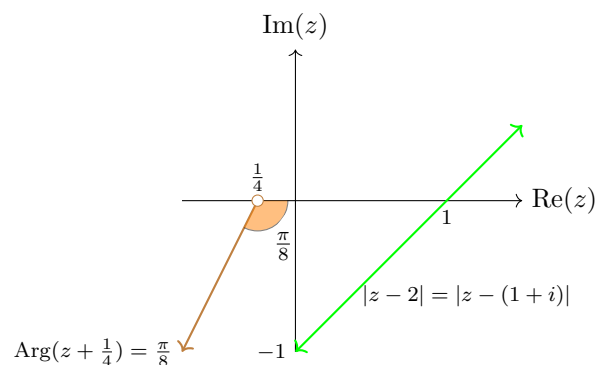
Argand planes



- Multiplication by $i \implies$ CCW rotation of $\frac{\pi}{2}$
- Addition: $z_1 + z_2 \equiv \vec{Oz_1} + \vec{Oz_2}$

Sketching complex graphs

Rays/lines $\text{Arg}(z \pm b) = \theta$

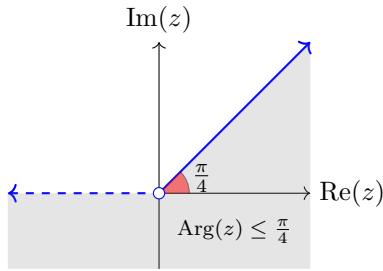


- $\text{Arg}(z \pm b) = \theta$ (ray)
- $\text{Re}(z) = c$ or $\text{Im}(z) = c$ (perpendicular bisector)
- $\text{Im}(z) = m \text{Re}(z)$
- $|z - (a + bi)| = |z - (c + di)| \implies \frac{2(c-a)x + a^2 + b^2 - c^2 - d^2}{2(b-d)}$
- $\text{Re}(z) \pm \text{Im}(z) = c$

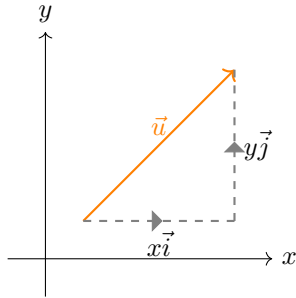
Circles

- $|z - z_1|^2 = c^2|z_2 + 2|^2$
- $|z - (a + bi)| = c \implies (x - a)^2 + (y - b)^2 = c^2$
- $z\bar{z} = r^2$

Regions $\text{Arg}(z) \leq \theta$



2 Vectors



Column notation

$$\begin{bmatrix} x \\ y \end{bmatrix} \iff xi + yj$$

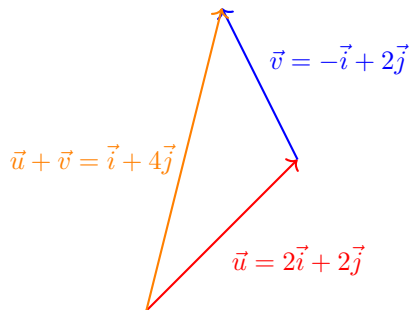
$$\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} \text{ between } A(x_1, y_1), B(x_2, y_2)$$

Scalar multiplication

$$k \cdot (xi + yj) = kxi + kyj$$

For $k \in \mathbb{R}^-$, direction is reversed

Vector addition



$$(xi + yj) \pm (ai + bj) = (x \pm a)i + (y \pm b)j$$

- Draw each vector head to tail then join lines
- Addition is commutative (parallelogram)
- $u - v = u + (-v) \implies \overrightarrow{AB} = b - a$

Magnitude

$$|(xi + yj)| = \sqrt{x^2 + y^2}$$

Parallel vectors

$$u \parallel v \iff u = kv \text{ where } k \in \mathbb{R} \setminus \{0\}$$

For parallel vectors a and b :

$$a \cdot b = \begin{cases} |a||b| & \text{if same direction} \\ -|a||b| & \text{if opposite directions} \end{cases}$$

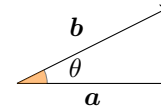
Perpendicular vectors

$$a \perp b \iff a \cdot b = 0 \quad (\text{since } \cos 90 = 0)$$

Unit vector $|\hat{a}| = 1$

$$\begin{aligned} \hat{a} &= \frac{1}{|a|} a \\ &= a \cdot |a| \end{aligned}$$

Scalar product $a \cdot b$



$$\begin{aligned} a \cdot b &= a_1b_1 + a_2b_2 \\ &= |a||b| \cos \theta \end{aligned}$$

$(0 \leq \theta \leq \pi)$ - from cosine rule

On CAS: dotP([a b c], [d e f])

Properties

1. $k(a \cdot b) = (ka) \cdot b = a \cdot (kb)$
2. $a \cdot 0 = 0$
3. $a \cdot (b + c) = a \cdot b + a \cdot c$
4. $i \cdot i = j \cdot j = k \cdot k = 1$
5. $a \cdot b = 0 \implies a \perp b$
6. $a \cdot a = |a|^2 = a^2$

Angle between vectors

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{a_1b_1 + a_2b_2}{|\mathbf{a}||\mathbf{b}|}$$

On CAS: angle([a b c], [a b c])

(Action → Vector → Angle)

Angle between vector and axis

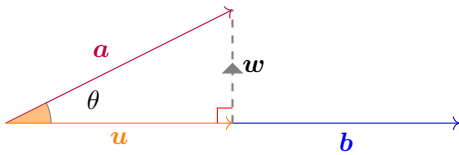
For $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ which makes angles α, β, γ with positive side of x, y, z axes:

$$\cos \alpha = \frac{a_1}{|\mathbf{a}|}, \quad \cos \beta = \frac{a_2}{|\mathbf{a}|}, \quad \cos \gamma = \frac{a_3}{|\mathbf{a}|}$$

On CAS: angle([a b c], [1 0 0])

for angle between $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and x -axis

Projections & resolute



$\parallel \mathbf{b}$ (vector projection/resolute)

$$\begin{aligned} \mathbf{u} &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2} \mathbf{b} \\ &= \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \right) \left(\frac{\mathbf{b}}{|\mathbf{b}|} \right) \\ &= (\mathbf{a} \cdot \hat{\mathbf{b}}) \hat{\mathbf{b}} \end{aligned}$$

$\perp \mathbf{b}$ (perpendicular projection)

$$\mathbf{w} = \mathbf{a} - \mathbf{u}$$

$|\mathbf{u}|$ (scalar projection/resolute)

$$\begin{aligned} s &= |\mathbf{u}| \\ &= \mathbf{a} \cdot \hat{\mathbf{b}} \\ &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \\ &= |\mathbf{a}| \cos \theta \end{aligned}$$

Rectangular (\parallel, \perp) components

$$\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b} + \left(\mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b} \right)$$

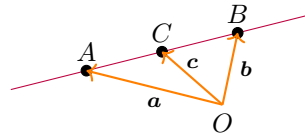
Vector proofs

Concurrent: intersection of ≥ 3 lines



Collinear points

≥ 3 points lie on the same line



e.g. Prove that

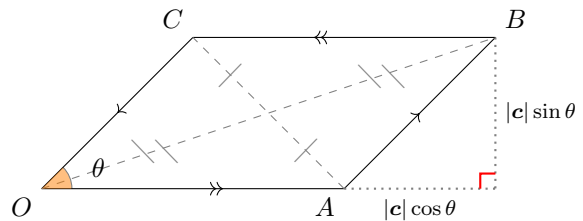
$$\begin{aligned} \vec{AC} = m\vec{AB} &\iff \mathbf{c} = (1 - m)\mathbf{a} + m\mathbf{b} \\ &\implies \mathbf{c} = \vec{OA} + \vec{AC} \\ &= \vec{OA} + m\vec{AB} \\ &= \mathbf{a} + m(\mathbf{b} - \mathbf{a}) \\ &= \mathbf{a} + m\mathbf{b} - m\mathbf{a} \\ &= (1 - m)\mathbf{a} + m\mathbf{b} \end{aligned}$$

$$\text{Also, } \implies \vec{OC} = \lambda\vec{OA} + \mu\vec{OB}$$

$$\text{where } \lambda + \mu = 1$$

$$\text{If } C \text{ lies along } \vec{AB}, \implies 0 < \mu < 1$$

Parallelograms



• Diagonals \vec{OB}, \vec{AC} bisect each other

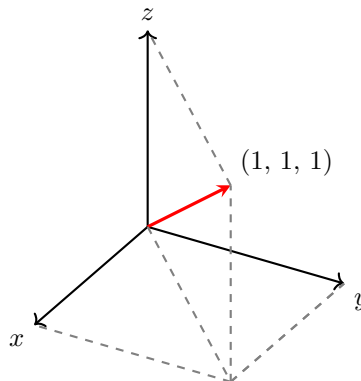
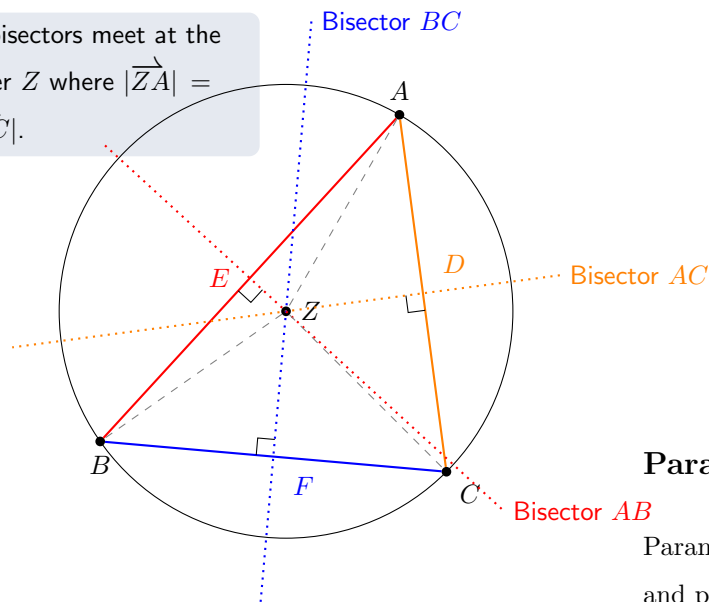
• If diagonals are equal length, it is a rectangle

$$\bullet |\vec{OB}|^2 + |\vec{CA}|^2 = |\vec{OA}|^2 + |\vec{AB}|^2 + |\vec{CB}|^2 + |\vec{OC}|^2$$

• Area = $\mathbf{c} \cdot \mathbf{a}$

Perpendicular bisectors of a triangle

The three bisectors meet at the circumcenter Z where $|\vec{ZA}| = |\vec{ZB}| = |\vec{ZC}|$.



Perpendicular bisector theorem

If a point P lies on the perpendicular bisector of line \vec{XY} , then P is equidistant from the endpoints of the bisected segment

i.e. $|\vec{PX}| = |\vec{PY}|$

Parametric vectors

Parametric equation of line through point (x_0, y_0, z_0) and parallel to $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is:

$$\begin{cases} x = x_0 + a \cdot t \\ y = y_0 + b \cdot t \\ z = z_0 + c \cdot t \end{cases}$$

Useful vector properties

- $\mathbf{a} \parallel \mathbf{b} \implies \mathbf{b} = k\mathbf{a}$ for some $k \in \mathbb{R} \setminus \{0\}$
- If \mathbf{a} and \mathbf{b} are parallel with at least one point in common, then they lie on the same straight line
- $\mathbf{a} \perp \mathbf{b} \iff \mathbf{a} \cdot \mathbf{b} = 0$
- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

Linear dependence

$\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly dependent if they are \nparallel and:

$$0 = k\mathbf{a} + l\mathbf{b} + m\mathbf{c}$$

$$\therefore \mathbf{c} = m\mathbf{a} + n\mathbf{b} \quad (\text{simultaneous})$$

\mathbf{a}, \mathbf{b} , and \mathbf{c} are linearly independent if no vector in the set is expressible as a linear combination of other vectors in set, or if they are parallel.

Three-dimensional vectors

Right-hand rule for axes: z is up or out of page.

3 Circular functions

$\sin(bx)$ or $\cos(bx)$: period = $\frac{2\pi}{b}$

$\tan(nx)$: period = $\frac{\pi}{n}$

asymptotes at $x = \frac{(2k+1)\pi}{2n} \mid k \in \mathbb{Z}$

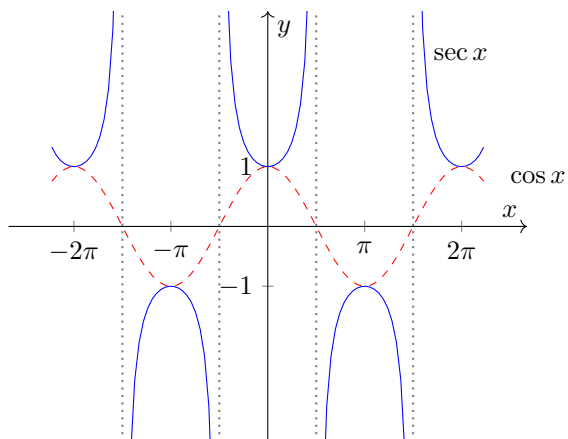
Reciprocal functions

Cosecant

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta} \mid \sin \theta \neq 0$$

- **Domain** = $\mathbb{R} \setminus n\pi : n \in \mathbb{Z}$
- **Range** = $\mathbb{R} \setminus (-1, 1)$
- **Turning points** at $\theta = \frac{(2n+1)\pi}{2} \mid n \in \mathbb{Z}$
- **Asymptotes** at $\theta = n\pi \mid n \in \mathbb{Z}$

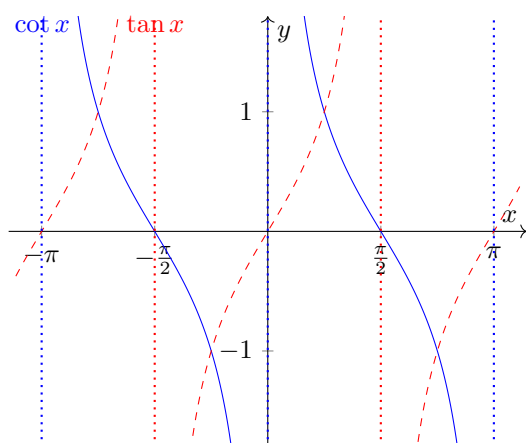
Secant



$$\sec \theta = \frac{1}{\cos \theta} \mid \cos \theta \neq 0$$

- **Domain** = $\mathbb{R} \setminus \left\{ \frac{(2n+1)\pi}{2} : n \in \mathbb{Z} \right\}$
- **Range** = $\mathbb{R} \setminus (-1, 1)$
- **Turning points** at $\theta = n\pi \mid n \in \mathbb{Z}$
- **Asymptotes** at $\theta = \frac{(2n+1)\pi}{2} \mid n \in \mathbb{Z}$

Cotangent



$$\cot \theta = \frac{\cos \theta}{\sin \theta} \mid \sin \theta \neq 0$$

- **Domain** = $\mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$
- **Range** = \mathbb{R}
- **Asymptotes** at $\theta = n\pi \mid n \in \mathbb{Z}$

Symmetry properties

$$\sec(\pi \pm x) = -\sec x$$

$$\sec(-x) = \sec x$$

$$\operatorname{cosec}(\pi \pm x) = \mp \operatorname{cosec} x$$

$$\operatorname{cosec}(-x) = -\operatorname{cosec} x$$

$$\cot(\pi \pm x) = \pm \cot x$$

$$\cot(-x) = -\cot x$$

Complementary properties

$$\sec\left(\frac{\pi}{2} - x\right) = \operatorname{cosec} x$$

$$\operatorname{cosec}\left(\frac{\pi}{2} - x\right) = \sec x$$

$$\cot\left(\frac{\pi}{2} - x\right) = \tan x$$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x$$

Pythagorean identities

$$1 + \cot^2 x = \operatorname{cosec}^2 x, \quad \text{where } \sin x \neq 0$$

$$1 + \tan^2 x = \sec^2 x, \quad \text{where } \cos x \neq 0$$

Compound angle formulas

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

Double angle formulas

$$\cos 2x = \cos^2 x - \sin^2 x$$

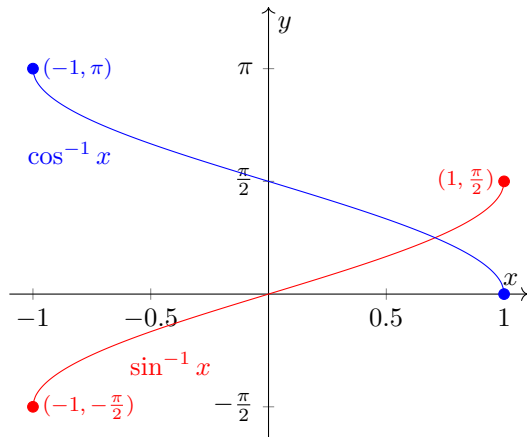
$$= 1 - 2\sin^2 x$$

$$= 2\cos^2 x - 1$$

$$\sin 2x = 2\sin x \cos x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Inverse circular functions



Segments: $A = \frac{r^2}{2} (\theta - \sin \theta)$

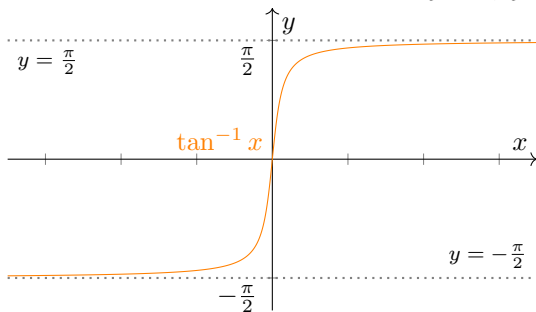
Chords: $\text{crd } \theta = \sqrt{(1 - \cos \theta)^2 + \sin^2 \theta}$
 $= \sqrt{2 - 2 \cos \theta}$
 $= 2 \sin \left(\frac{\theta}{2} \right)$

Inverse functions: $f(f^{-1}(x)) = x$ (restrict domain)

$\sin^{-1} : [-1, 1] \rightarrow \mathbb{R}, \quad \sin^{-1} x = y$
 where $\sin y = x, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\cos^{-1} : [-1, 1] \rightarrow \mathbb{R}, \quad \cos^{-1} x = y$
 where $\cos y = x, y \in [0, \pi]$

$\tan^{-1} : \mathbb{R} \rightarrow \mathbb{R}, \quad \tan^{-1} x = y$
 where $\tan y = x, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



4 Differential calculus

$f'(x) = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$

Limits

$\lim_{x \rightarrow a} f(x)$

L^-, L^+ limit from below/above

$\lim_{x \rightarrow a} f(x)$ limit of a point

For solving $x \rightarrow \infty$, put all x terms in denominators

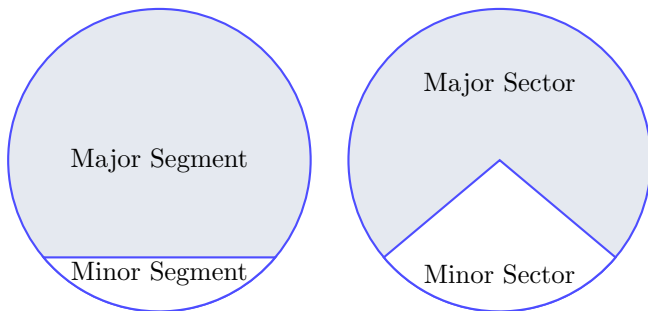
e.g.

$\lim_{x \rightarrow \infty} \frac{2x + 3}{x - 2} = \frac{2 + \frac{3}{x}}{1 - \frac{2}{x}} = \frac{2}{1} = 2$

Limit theorems

1. For constant function $f(x) = k, \lim_{x \rightarrow a} f(x) = k$
2. $\lim_{x \rightarrow a} (f(x) \pm g(x)) = F \pm G$
3. $\lim_{x \rightarrow a} (f(x) \times g(x)) = F \times G$
4. $\therefore \lim_{x \rightarrow a} c \times f(x) = cF$ where $c = \text{constant}$
5. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{F}{G}, G \neq 0$
6. $f(x)$ is continuous $\iff L^- = L^+ = f(x) \forall x$

Mensuration



Sectors: $A = \pi r^2 \frac{\theta}{2\pi}$
 $= \frac{r^2 \theta}{2}$

Gradients

Secant (chord) - line joining two points on curve

Tangent - line that intersects curve at one point

Points of Inflection

Stationary point - i.e. $f'(x) = 0$

Point of inflection - max |gradient| (i.e. $f'' = 0$)

Strictly increasing/decreasing

For x_2 and x_1 where $x_2 > x_1$:

strictly increasing

where $f(x_2) > f(x_1)$ or $f'(x) > 0$

strictly decreasing

where $f(x_2) < f(x_1)$ or $f'(x) < 0$

Endpoints are included, even where $\frac{dy}{dx} = 0$

If p and q are expressions in x and y such that $p = q$, for all x and y , then:

$$\frac{dp}{dx} = \frac{dq}{dx} \quad \text{and} \quad \frac{dp}{dy} = \frac{dq}{dy}$$

On CAS

Action → Calculation

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impDiff(y^2+ax=5, x, y)
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Second derivative

$$f(x) \rightarrow f'(x) \rightarrow f''(x)$$

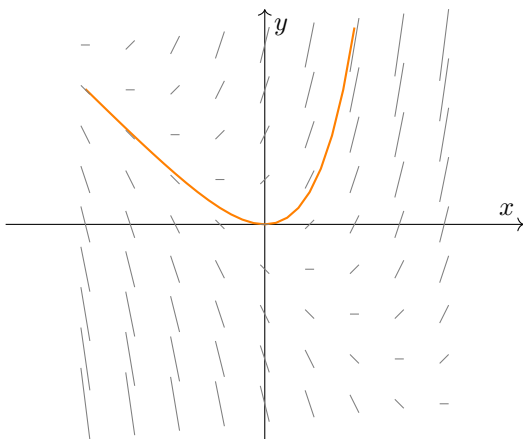
$$\Rightarrow y \rightarrow \frac{dy}{dx} \rightarrow \frac{d^2y}{dx^2}$$

Order of polynomial n th derivative decrements each time the derivative is taken

Function of the dependent variable

If $\frac{dy}{dx} = g(y)$, then $\frac{dx}{dy} = 1 \div \frac{dy}{dx} = \frac{1}{g(y)}$. Integrate both sides to solve equation. Only add c on one side. Express e^c as A .

Slope fields



- $f'(a) = 0, f''(a) > 0$
local min at $(a, f(a))$ (concave up)
- $f'(a) = 0, f''(a) < 0$
local max at $(a, f(a))$ (concave down)
- $f''(a) = 0$
point of inflection at $(a, f(a))$
- $f''(a) = 0, f'(a) = 0$
 stationary point of inflection at $(a, f(a))$

Reciprocal derivatives

$$\frac{1}{\frac{dy}{dx}} = \frac{dx}{dy}$$

Differentiating $x = f(y)$

Find $\frac{dx}{dy}$, then $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$

Parametric equations

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \text{ provided } \frac{dx}{dt} \neq 0$$

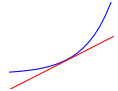
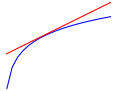
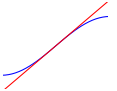
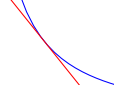
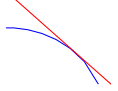
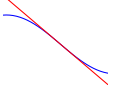
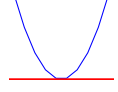

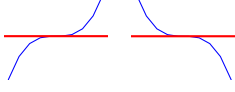
$$\frac{d^2y}{dx^2} = \frac{\left(\frac{dy'}{dt}\right)}{\left(\frac{dx}{dt}\right)} \text{ where } y' = \frac{dy}{dx}$$

Implicit Differentiation

Used for differentiating circles etc.

Integration

$$\int f(x) \cdot dx = F(x) + c \quad \text{where } F'(x) = f(x)$$

	$\frac{d^2y}{dx^2} > 0$	$\frac{d^2y}{dx^2} < 0$	$\frac{d^2y}{dx^2} = 0$ (inflection)
$\frac{dy}{dx} > 0$	 Rising (concave up)	 Rising (concave down)	 Rising inflection point
$\frac{dy}{dx} < 0$	 Falling (concave up)	 Falling (concave down)	 Falling inflection point
$\frac{dy}{dx} = 0$	 Local minimum	 Local maximum	 Stationary inflection point

Properties

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$\int_a^a f(x) dx = 0$$

$$\int_a^b k \cdot f(x) dx = k \int_a^b f(x) dx$$

$$\int_a^b f(x) \pm g(x) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

Integration by substitution

$$\int f(u) \frac{du}{dx} \cdot dx = \int f(u) \cdot du$$

$f(u)$ must be 1:1 \implies one x for each y

e.g. for $y = \int (2x + 1)\sqrt{x + 4} \cdot dx$

let $u = x + 4$

$\implies \frac{du}{dx} = 1$

$\implies x = u - 4$

then $y = \int (2(u - 4) + 1)u^{\frac{1}{2}} \cdot du$

(solve as normal integral)

Definite integrals by substitution

For $\int_a^b f(x) \frac{du}{dx} \cdot dx$, evaluate new a and b for $f(u) \cdot du$.

Trigonometric integration

$$\sin^m x \cos^n x \cdot dx$$

m is odd: $m = 2k + 1$ where $k \in \mathbb{Z}$

$\implies \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x$

Substitute $u = \cos x$

n is odd: $n = 2k + 1$ where $k \in \mathbb{Z}$

$\implies \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x$

Substitute $u = \sin x$

m and n are even: use identities...

- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

- $\sin 2x = 2 \sin x \cos x$

Separation of variables

If $\frac{dy}{dx} = f(x)g(y)$, then:

$$\int f(x) dx = \int \frac{1}{g(y)} dy$$

Partial fractions

To factorise $f(x) = \frac{\delta}{\alpha \cdot \beta}$:

$$\frac{\delta}{\alpha \cdot \beta \cdot \gamma} = \frac{A}{\alpha} + \frac{B}{\beta} + \frac{C}{\gamma} \tag{1}$$

Multiply by $(\alpha \cdot \beta \cdot \gamma)$:

$$\delta = \beta\gamma A + \alpha\gamma B + \alpha\beta C \tag{2}$$

Substitute $x = \{\alpha, \beta, \gamma\}$ into (2) to find denominators

Repeated linear factors

$$\frac{p(x)}{(x-a)^n} = \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_n}{(x-a)^n}$$

Irreducible quadratic factors

e.g. $\frac{3x-4}{(2x-3)(x^2+5)} = \frac{A}{2x-3} + \frac{Bx+C}{x^2+5}$

On CAS

Action → Transformation:
`expand(..., x)`

To reverse, use `combine(...)`

Integrating $\frac{dy}{dx} = g(y)$

if $\frac{dy}{dx} = g(y)$, then $x = \int \frac{1}{g(y)} dy$

Graphing integrals on CAS

On CAS

In main: Interactive → Calculation → \int

For restrictions, `Define f(x)=...` then `f(x)|x>...`

Solids of revolution

Approximate as sum of infinitesimally-thick cylinders

Rotation about x-axis

$$V = \pi \int_{x=a}^{x=b} f(x)^2 dx$$

Rotation about y-axis

$$V = \pi \int_{y=a}^{y=b} x^2 dy$$

$$= 2\pi \int_{x=a}^{x=b} x|f(x)| dx$$

Rotating the area between two graphs

$$V = \pi \int_a^b (f(x)^2 - g(x)^2) dx$$

where $f(x) > g(x)$

Length of a curve

For length of $f(x)$ from $x = a \rightarrow x = b$:

Cartesian $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

Parametric $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

On CAS

a) Evaluate formula

b) Interactive → Calculation → Line → `arcLen`

Applications of antidifferentiation

- x -intercepts of $y = f(x)$ identify x -coordinates of stationary points on $y = F(x)$
- nature of stationary points is determined by sign of $y = f(x)$ on either side of its x -intercepts
- if $f(x)$ is a polynomial of degree n , then $F(x)$ has degree $n + 1$

To find stationary points of a function, substitute x value of given point into derivative. Solve for $\frac{dy}{dx} = 0$. Integrate to find original function.

Rates

Gradient at a point on parametric curve

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} \Big| \frac{dx}{dt} \neq 0 \text{ (chain rule)}$$

$$\frac{d^2}{dx^2} = \frac{d(y')}{dx} = \frac{dy'}{dt} \div \frac{dx}{dt} \Big| y' = \frac{dy}{dx}$$

Rational functions

$$f(x) = \frac{P(x)}{Q(x)} \text{ where } P, Q \text{ are polynomial functions}$$

Euler's method

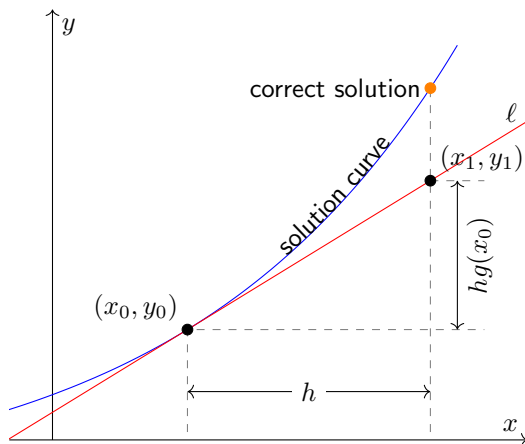
$$\frac{f(x+h) - f(x)}{h} \approx f'(x) \quad \text{for small } h$$

$$\implies f(x+h) \approx f(x) + hf'(x)$$

If $\frac{dy}{dx} = g(x)$ with $x_0 = a$ and $y_0 = b$, then:

$$\begin{cases} x_{n+1} = x_n + h \\ y_{n+1} = y_n + hg(x_n) \end{cases}$$

$$\frac{d^2y}{dx^2} \begin{cases} > 0 \implies \text{underestimate (concave up)} \\ < 0 \implies \text{overestimate (concave down)} \end{cases}$$



On CAS

Menu → Sequence → Recursive

To generate x-values:

- Enter $a_{n+1} = a_n + h$ where h is the step size (input a_n from menu bar)
- In a_0 , set the initial value x_0 as a constant

To generate y-values:

- In b_{n+1} , enter $\frac{dy}{dx}$, replacing x with a_n
- Set $b_0 = y(x_0)$ as a constant

To view table of values, tap table icon (top left)

To compare approximations with actual values, enter in $c_{n+1} = a_{n+1} - f(a_{n+1})$ where $f(x) = \int \frac{dy}{dx} dx$

Fundamental theorem of calculus

If f is continuous on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

where $F = \int f dx$

Differential equations

Order - highest power inside derivative

Degree - highest power of highest derivative

e.g. $\left(\frac{dy^2}{dx^2} x\right)^3$ order 2, degree 3

To verify solutions, find $\frac{dy}{dx}$ from y and substitute into original

DE	Method
$\frac{dy}{dx} = f(x)$	$y = \int f(x) dx$ $= F(x) + c$ where $F'(x) = f(x)$
$\frac{d^2y}{dx^2} = f(x)$	$\frac{dy}{dx} = \int f(x) dx$ $= F(x) + c$ where $F'(x) = f(x)$ $\therefore y = \iint f(x) dx = \int (F(x) + c) dx$ $= G(x) + cx + d$ where $G'(x) = F(x)$
$\frac{dy}{dx} = g(y)$	$\frac{dx}{dy} = \frac{1}{g(y)}$ $\therefore x = \int \frac{1}{g(y)} dy$ $= F(y) + c$ where $F'(y) = \frac{1}{g(y)}$
$\frac{dy}{dx} = f(x)g(y)$	$f(x) = \frac{1}{g(y)} \cdot \frac{dy}{dx}$ $\int f(x) dx = \int \frac{1}{g(y)} dy$

Mixing problems

$$\left(\frac{dm}{dt}\right)_{\Sigma} = \left(\frac{dm}{dt}\right)_{\text{in}} - \left(\frac{dm}{dt}\right)_{\text{out}}$$

Derivatives

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\sin ax$	$a \cos ax$
$\cos x$	$-\sin x$
$\cos ax$	$-a \sin ax$
$\tan f(x)$	$f^2(x) \sec^2 f(x)$
e^x	e^x
e^{ax}	ae^{ax}
ax^{nx}	$an \cdot e^{nx}$
$\log_e x$	$\frac{1}{x}$
$\log_e ax$	$\frac{1}{x}$
$\log_e f(x)$	$\frac{f'(x)}{f(x)}$
$\sin(f(x))$	$f'(x) \cdot \cos(f(x))$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\frac{d}{dy} f(y)$	$\frac{1}{\frac{dx}{dy}}$ (reciprocal)
uv	$u \frac{dv}{dx} + v \frac{du}{dx}$ (product rule)
$\frac{u}{v}$	$\frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$ (quotient rule)
$f(g(x))$	$f'(g(x)) \cdot g'(x)$

Antiderivatives

$f(x)$	$\int f(x) \cdot dx$
k (constant)	$kx + c$
x^n	$\frac{1}{n+1} x^{n+1}$
ax^{-n}	$a \cdot \log_e x + c$
$\frac{1}{ax+b}$	$\frac{1}{a} \log_e(ax+b) + c$
$(ax+b)^n$	$\frac{1}{a(n+1)} (ax+b)^{n+1} + c \mid n \neq -1$
$(ax+b)^{-1}$	$\frac{1}{a} \log_e ax+b + c$
e^{kx}	$\frac{1}{k} e^{kx} + c$
e^k	$e^k x + c$
$\sin kx$	$-\frac{1}{k} \cos(kx) + c$
$\cos kx$	$\frac{1}{k} \sin(kx) + c$
$\sec^2 kx$	$\frac{1}{k} \tan(kx) + c$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1} \frac{x}{a} + c \mid a > 0$
$\frac{-1}{\sqrt{a^2-x^2}}$	$\cos^{-1} \frac{x}{a} + c \mid a > 0$
$\frac{a}{a^2-x^2}$	$\tan^{-1} \frac{x}{a} + c$
$\frac{f'(x)}{f(x)}$	$\log_e f(x) + c$
$\int f(u) \cdot \frac{du}{dx} \cdot dx$	$\int f(u) \cdot du$ (substitution)
$f(x) \cdot g(x)$	$\int [f'(x) \cdot g(x)] dx + \int [g'(x) f(x)] dx$

Note $\sin^{-1} \left(\frac{x}{a} \right) + \cos^{-1} \left(\frac{x}{a} \right)$ is constant $\forall x \in (-a, a)$

Index identities

$$a^{x+y} = a^x \cdot a^y$$

$$a^{x-y} = a^x \div a^y$$

$$(a^x)^y = a^{x \cdot y}$$

$$(a \cdot b)^x = a^x \cdot b^x$$

Logarithmic identities

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b \left(\frac{x}{y} \right) = \log_b(x) - \log_b(y)$$

$$\log_b y^{x^n} = x^n \log_b y$$

$$\log_b x^n = n \log_b x$$

5 Kinematics & Mechanics

Constant acceleration

- **Position** - relative to origin
- **Displacement** - relative to starting point

Velocity-time graphs

Displacement: *signed* area

Distance travelled: *total* area

$$\text{acceleration} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v \frac{dv}{dx} = \frac{d}{dx} \left(\frac{1}{2}v^2 \right)$$

	no
$v = u + at$	x
$v^2 = u^2 + 2as$	t
$s = \frac{1}{2}(v + u)t$	a
$s = ut + \frac{1}{2}at^2$	v
$s = vt - \frac{1}{2}at^2$	u

$$v_{\text{avg}} = \frac{\Delta \text{position}}{\Delta t}$$

$$\begin{aligned} \text{speed} &= |\text{velocity}| \\ &= \sqrt{v_x^2 + v_y^2 + v_z^2} \end{aligned}$$

Distance travelled between $t = a \rightarrow t = b$:

$$\begin{aligned} &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt && (2D) \\ &= \int_{t=a}^{t=b} \frac{dx}{dt} dt && (\text{linear}) \end{aligned}$$

Shortest distance between $\mathbf{r}(t_0)$ and $\mathbf{r}(t_1)$:

$$= |\mathbf{r}(t_1) - \mathbf{r}(t_2)|$$

Vector functions

$$\mathbf{r}(t) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

- If $\mathbf{r}(t) \equiv$ position with time, then the graph of endpoints of $\mathbf{r}(t) \equiv$ Cartesian path
- Domain of $\mathbf{r}(t)$ is the range of $x(t)$
- Range of $\mathbf{r}(t)$ is the range of $y(t)$

Vector calculus

Derivative

Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. If both $x(t)$ and $y(t)$ are differentiable, then:

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

6 Dynamics

Resolution of forces

Resultant force is sum of force vectors

In angle-magnitude form

$$\begin{aligned} \text{Cosine rule:} \quad c^2 &= a^2 + b^2 - 2ab \cos \theta \\ \text{Sine rule:} \quad \frac{a}{\sin A} &= \frac{b}{\sin B} = \frac{c}{\sin C} \end{aligned}$$

In $i-j$ form

Vector of a N at θ to x axis is equal to $a \cos \theta \mathbf{i} + a \sin \theta \mathbf{j}$.

Convert all force vectors then add.

To find angle of an $a\mathbf{i} + b\mathbf{j}$ vector, use $\theta = \tan^{-1} \frac{b}{a}$

Resolving in a given direction

The resolved part of a force P at angle θ is has magnitude $P \cos \theta$

To convert force $\|\vec{OA}$ to angle-magnitude form, find component $\perp \vec{OA}$ then:

$$\begin{aligned} |\mathbf{r}| &= \sqrt{\left(\|\vec{OA}\right)^2 + \left(\perp \vec{OA}\right)^2} \\ \theta &= \tan^{-1} \frac{\perp \vec{OA}}{\|\vec{OA}\}} \end{aligned}$$

Newton's laws

1. Velocity is constant without ΣF
2. $\frac{d}{dt}\rho \propto \Sigma F \implies \mathbf{F} = m\mathbf{a}$
3. Equal and opposite forces

Weight

A mass of m kg has force of mg acting on it

Momentum ρ

$$\rho = mv \quad (\text{units kg m/s or Ns})$$

Reaction force R

- With no vertical velocity, $R = mg$
- With vertical acceleration, $|R| = m|a| - mg$
- With force F at angle θ , then $R = mg - F \sin \theta$

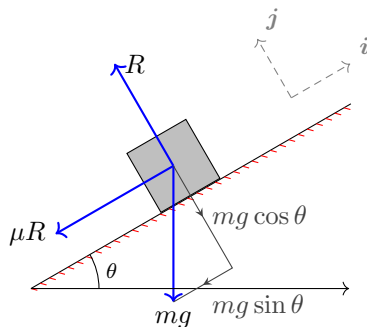
Friction

$$F_R = \mu R \quad (\text{friction coefficient})$$

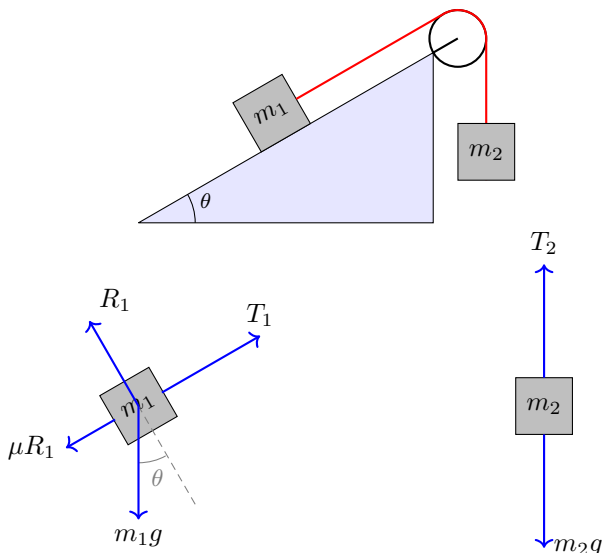
Inclined planes

$$\mathbf{F} = |\mathbf{F}| \cos \theta \mathbf{i} + |\mathbf{F}| \sin \theta \mathbf{j}$$

- Normal force R is at right angles to plane
- Let direction up the plane be \mathbf{i} and perpendicular to plane \mathbf{j}



Connected particles



- **Suspended pulley:** $T_1 = T_2$

$$|a| = g \frac{m_1 - m_2}{m_1 + m_2} \text{ where } m_1 \text{ accelerates down}$$

$$\begin{cases} m_1g - T = m_1a \\ T - m_2g = m_2a \end{cases} \implies m_1g - m_2g = m_1a + m_2a$$

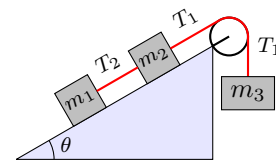
- **String pulling mass on inclined pane:** Resolve parallel to plane

$$T - mg \sin \theta = ma$$

- **Linear connection:** find acceleration of system first

- **Pulley on right angle:** $a = \frac{m_2g}{m_1+m_2}$ where m_2 is suspended (frictionless on both surfaces)

- **Pulley on edge of incline:** find downwards force W_2 and components of mass on plane



In this example, note $T_1 \neq T_2$:

Equilibrium

$$\frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c} \quad (\text{Lami's theorem})$$

$$c^2 = a^2 + b^2 - 2ab \cos \theta \quad (\text{cosine rule})$$

Three methods:

1. Lami's theorem (sine rule)
2. Triangle of forces (cosine rule)
3. Resolution of forces ($\Sigma F = 0$ - simultaneous)

On CAS

To verify: Geometry tab, then select points with normal cursor. Click right arrow at end of toolbar and input point, then lock known constants.

Variable forces (DEs)

$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v \frac{dv}{dx} = \frac{d}{dx} \left(\frac{1}{2}v^2 \right)$$

7 Statistics

Continuous random variables

A continuous random variable X has a pdf f such that:

1. $f(x) \geq 0 \forall x$

2. $\int_{-\infty}^{\infty} f(x) dx = 1$

$$E(X) = \int_{\mathbf{x}} (x \cdot f(x)) dx$$

$$\text{Var}(X) = E[(X - \mu)^2]$$

$$\text{Pr}(X \leq c) = \int_{-\infty}^c f(x) dx$$

Two random variables X, Y

If X and Y are independent:

$$E(aX + bY) = aE(X) + bE(Y)$$

$$\text{Var}(aX \pm bY \pm c) = a^2 \text{Var}(X) + b^2 \text{Var}(Y)$$

Linear functions $X \rightarrow aX + b$

$$\begin{aligned} \text{Pr}(Y \leq y) &= \text{Pr}(aX + b \leq y) \\ &= \text{Pr}\left(X \leq \frac{y-b}{a}\right) \\ &= \int_{-\infty}^{\frac{y-b}{a}} f(x) dx \end{aligned}$$

Mean: $E(aX + b) = aE(X) + b$

Variance: $\text{Var}(aX + b) = a^2 \text{Var}(X)$

Expectation theorems

For some non-linear function g , the expected value $E(g(X))$ is not equal to $g(E(X))$.

$$E(X^2) = \text{Var}(X) + [E(X)]^2$$

$$E(X^n) = \sum x^n \cdot p(x) \quad (\text{non-linear})$$

$$\neq [E(X)]^n$$

$$E(aX \pm b) = aE(X) \pm b \quad (\text{linear})$$

$$E(b) = b \quad (\forall b \in \mathbb{R})$$

$$E(X + Y) = E(X) + E(Y) \quad (\text{two variables})$$

Sample mean

Approximation of the **population mean** determined experimentally.

$$\bar{x} = \frac{\sum x}{n}$$

where

n is the size of the sample (number of sample points)

x is the value of a sample point

On CAS

1. Spreadsheet
2. In cell A1:
`mean(randNorm(sd, mean, sample size))`
3. Edit → Fill → Fill Range
4. Input range as A1:An where n is the number of samples
5. Graph → Histogram

Sample size of n

$$\bar{X} = \sum_{i=1}^n \frac{x_i}{n} = \frac{\sum x}{n}$$

Sample mean is distributed with mean μ and sd $\frac{\sigma}{\sqrt{n}}$ (approaches these values for increasing sample size n).

For a new distribution with mean of n trials, $E(X') = E(X)$, $\text{sd}(X') = \frac{\text{sd}(X)}{\sqrt{n}}$

On CAS

Spreadsheet → Catalog →
`randNorm(sd, mean, n)`
 where n is the number of samples. Show histogram
 with Histogram key in top left.
 To calculate parameters of a dataset:
 Calc → One-variable

\bar{x} is the sample mean

σ is the population sd

n is the sample size from which \bar{x} was calculated

On CAS

Menu → Stats → Calc → Interval
 Set *Type = One-Sample Z Int*
 and select *Variable*

Normal distributions

$$Z = \frac{X - \mu}{\sigma}$$

Normal distributions must have area (total prob.) of 1

$$\implies \int_{-\infty}^{\infty} f(x) dx = 1$$

mean = mode = median

Always express z as +ve. Express confidence interval as ordered pair.

Central limit theorem

If X is randomly distributed with mean μ and sd σ , then with an adequate sample size n the distribution of the sample mean \bar{X} is approximately normal with mean $E(\bar{X})$ and $\text{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}}$.

Confidence intervals

- **Point estimate:** single-valued estimate of the population mean from the value of the sample mean \bar{x}
- **Interval estimate:** confidence interval for population mean μ
- $C\%$ confidence interval $\implies C\%$ of samples will contain population mean μ

95% confidence interval

For 95% c.i. of population mean μ :

$$x \in \left(\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}} \right)$$

where:

Margin of error

For 95% confidence interval of μ :

$$M = 1.96 \times \frac{\sigma}{\sqrt{n}}$$

$$\implies n = \left(\frac{1.96\sigma}{M} \right)^2$$

Always round n up to a whole number of samples.

General case

For $C\%$ c.i. of population mean μ :

$$x \in \left(\bar{x} \pm k \frac{\sigma}{\sqrt{n}} \right)$$

where k is such that $\Pr(-k < Z < k) = \frac{C}{100}$

Confidence interval for multiple trials

For a set of n confidence intervals (samples), there is 0.95^n chance that all n intervals contain the population mean μ .

8 Hypothesis testing

Note hypotheses are always expressed in terms of population parameters

Null hypothesis H_0

Sample drawn from population has same mean as control population, and any difference can be explained by sample variations.

Alternative hypothesis H_1

Amount of variation from control is significant, despite standard sample variations.

If $p < \alpha$, null hypothesis is **rejected**

If $p > \alpha$, null hypothesis is **accepted**

p -value

Probability of observing a value of the sample statistic as significant as the one observed, assuming null hypothesis is true.

For one-tail tests:

$$p\text{-value} = \Pr(\bar{X} \leq \mu(H_1) \mid \mu = \mu(H_0))$$

$$= \Pr\left(Z \leq \frac{(\mu(H_1) - \mu(H_0)) \cdot \sqrt{n}}{\text{sd}(X)}\right)$$

then use `normCdf` with `std. norm.`

p	Conclusion
> 0.05	insufficient evidence against H_0
< 0.05 (5%)	good evidence against H_0
< 0.01 (1%)	strong evidence against H_0
< 0.001 (0.1%)	very strong evidence against H_0

Finding n for a given p -value

Find c such that $\Pr(Z \leq c)$ such that $c = \alpha$ (use `invNormCdf` on CAS).

Significance level α

The condition for rejecting the null hypothesis.

z -test

Hypothesis test for a mean of a sample drawn from a normally distributed population with a known standard deviation.

On CAS

Menu \rightarrow Statistics \rightarrow Calc \rightarrow Test.

Select *One-Sample Z-Test* and *Variable*, then input:

- μ cond: same operator as H_1
- μ_0 : expected sample mean (null hypothesis)
- σ : standard deviation (null hypothesis)
- \bar{x} : sample mean
- n : sample size

One-tail and two-tail tests

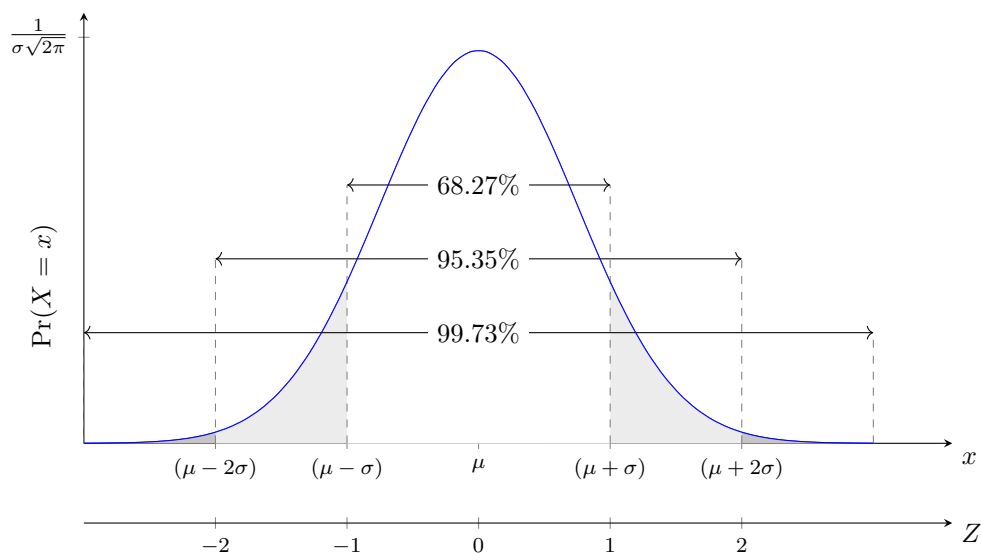
$$p\text{-value (two-tail)} = 2 \times p\text{-value (one-tail)}$$

One tail

- μ has changed in one direction
- State " $H_1 : \mu \leq$ known population mean"

Two tail

- Direction of $\Delta\mu$ is ambiguous



- State “ $\mathbf{H}_1 : \mu \neq$ known population mean”

$$\begin{aligned} p\text{-value} &= \Pr(|\bar{X} - \mu| \geq |\bar{x}_0 - \mu|) \\ &= \left(|Z| \geq \left| \frac{\bar{x}_0 - \mu}{\sigma \div \sqrt{n}} \right| \right) \end{aligned}$$

where

μ is the population mean under \mathbf{H}_0

\bar{x}_0 is the observed sample mean

σ is the population s.d.

n is the sample size

Modulus notation for two tail

$\Pr(|\bar{X} - \mu| \geq a) \implies$ “the probability that the distance between $\bar{\mu}$ and μ is $\geq a$ ”

Inverse normal

On CAS

```
invNormCdf("L",  $\alpha$ ,  $\frac{\sigma}{\sqrt{n}}$ ,  $\mu$ )
```

Errors

Type I error \mathbf{H}_0 is rejected when it is **true**

Type II error \mathbf{H}_0 is **not** rejected when it is **false**

	Actual result	
z -test	\mathbf{H}_0 true	\mathbf{H}_0 false
Reject \mathbf{H}_0	Type I error	Correct
Do not reject \mathbf{H}_0	Correct	Type II error