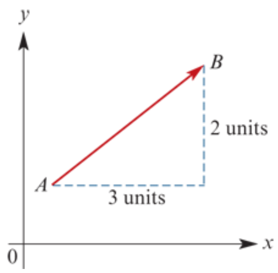
Vectors

- **vector:** a directed line segment

- arrow indicates direction
- length indicates magnitude
- notated as $\vec{a}, \widetilde{A}, a$
- column notation: $\begin{bmatrix} x \\ y \end{bmatrix}$
- vectors with equal magnitude and direction are equivalent



Vector addition

$\boldsymbol{u} + \boldsymbol{v}$ can be represented by drawing each vector head to tail then joining the lines.
Addition is commutative (parallelogram)

Scalar multiplication

For $k \in \mathbb{R}^+$, $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$ but length is multiplied by a factor of k.

When multiplied by $k < 0$, direction is reversed and length is multplied by k.

Vector subtraction

To find $\boldsymbol{u} - \boldsymbol{v}$, add $-\boldsymbol{v}$ to $\boldsymbol{u}$

Parallel vectors

Same or opposite direction

$$\boldsymbol{u} || \boldsymbol{v} \iff \boldsymbol{u} = k\boldsymbol{v} \text{ where } k \in \mathbb{R} \setminus \{0\}$$

Position vectors

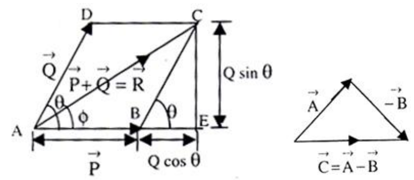
Vectors may describe a position relative to O.

For a point A, the position vector is $\overrightarrow{OA}$

Linear combinations of non-parallel vectors

If two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel, then:

$$m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b} \quad \therefore \quad m = p, \; n = q$$



Column vector notation

A vector between points $A(x_1, y_1)$, $B(x_2, y_2)$ can be represented as $\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$

Component notation

A vector $\boldsymbol{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ can be written as $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$.
$\boldsymbol{u}$ is the sum of two components $x\boldsymbol{i}$ and $y\boldsymbol{j}$
Magnitude of vector $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$ is denoted by $|u| = \sqrt{x^2 + y^2}$

Basic algebra applies:
$(x\boldsymbol{i} + y\boldsymbol{j}) + (m\boldsymbol{i} + n\boldsymbol{j}) = (x + m)\boldsymbol{i} + (y + n)\boldsymbol{j}$
Two vectors equal if and only if their components are equal.

Unit vector $|\hat{\boldsymbol{a}}| = 1$

$$\begin{aligned} \hat{\boldsymbol{a}} &= \frac{1}{|\boldsymbol{a}|}\boldsymbol{a} \\ &= \boldsymbol{a} \cdot |\boldsymbol{a}| \end{aligned} \tag{1}$$

Scalar/dot product $\boldsymbol{a} \cdot \boldsymbol{b}$

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$

on CAS: `dotP([a b c], [d e f])`

Scalar product properties

1. $k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (kb)$
2. $\boldsymbol{a} \cdot \boldsymbol{0} = 0$
3. $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
4. $\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1$
5. If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular
6. $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2 = a^2$

For parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{cases} |\boldsymbol{a}||\boldsymbol{b}| & \text{if same direction} \\ -|\boldsymbol{a}||\boldsymbol{b}| & \text{if opposite directions} \end{cases}$$

Geometric scalar products

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$$

where $0 \leq \theta \leq \pi$

Perpendicular vectors

If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, then $\boldsymbol{a} \perp \boldsymbol{b}$ (since $\cos 90 = 0$)

Finding angle between vectors

positive direction

$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{a_1 b_1 + a_2 b_2}{|\boldsymbol{a}||\boldsymbol{b}|}$$

on CAS: `angle([a b c], [a b c])` (Action -> Vector -> Angle)

Angle between vector and axis

Direction of a vector can be given by the angles it makes with $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ directions.

For $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ which makes angles α, β, γ with positive direction of x, y, z axes:

$$\cos\alpha = \frac{a_1}{|\boldsymbol{a}|}, \quad \cos\beta = \frac{a_2}{|\boldsymbol{a}|}, \quad \cos\gamma = \frac{a_3}{|\boldsymbol{a}|}$$

on CAS: `angle([a b c], [1 0 0])` for angle between $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ and x-axis

Vector projections

Vector resolute of $\boldsymbol{a}$ in direction of $\boldsymbol{b}$ is magnitude of $\boldsymbol{a}$ in direction of $\boldsymbol{b}$:

$$\boldsymbol{u} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|^2}\boldsymbol{b} = \left(\boldsymbol{a} \cdot \frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right)\left(\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right) = (\boldsymbol{a} \cdot \hat{\boldsymbol{b}})\hat{\boldsymbol{b}}$$

Scalar resolute of $\boldsymbol{a}$ on $\boldsymbol{b}$

$$r_s = |\boldsymbol{u}| = \boldsymbol{a} \cdot \hat{\boldsymbol{b}}$$

Vector resolute of $\boldsymbol{a} \perp \boldsymbol{b}$

$$\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u} \text{ where } \boldsymbol{u} \text{ is projection } \boldsymbol{a} \text{ on } \boldsymbol{b}$$

Vector proofs

Concurrent lines

≥ 3 lines intersect at a single point

Collinear points

≥ 3 points lie on the same line
$\implies \vec{OC} = \lambda\vec{OA} + \mu\vec{OB}$ where $\lambda + \mu = 1$. If C is between $\vec{AB}$, then $0 < \mu < 1$
Points A, B, C are collinear iff $\vec{AC} = m\vec{AB}$ where $m \neq 0$

Useful vector properties

- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then $\boldsymbol{b} = k\boldsymbol{a}$ for some $k \in \mathbb{R} \setminus \{0\}$
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel with at least one point in common, then they lie on the same straight line
- Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$

Linear dependence

Vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are linearly dependent if they are non-parallel and:

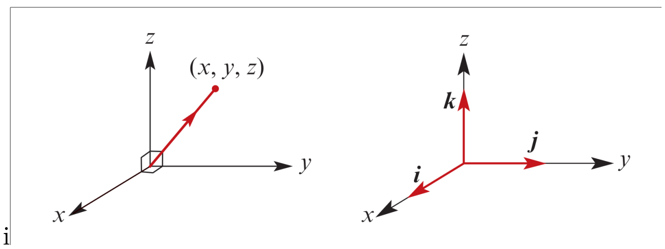
$$k\boldsymbol{a} + l\boldsymbol{b} + m\boldsymbol{c} = 0$$
$$\therefore \boldsymbol{c} = m\boldsymbol{a} + n\boldsymbol{b} \quad \text{(simultaneous)}$$

$\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$ are linearly independent if no vector in the set is expressible as a linear combination of other vectors in set, or if they are parallel.

Vector $\boldsymbol{w}$ is a linear combination of vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$

Three-dimensional vectors

Right-hand rule for axes: z is up or out of page.



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Parametric vectors

Parametric equation of line through point (x_0, y_0, z_0) and parallel to $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ is:

$$\begin{cases} x = x_o + a \cdot t \\ y = y_0 + b \cdot t \\ z = z_0 + c \cdot t \end{cases} \tag{2}$$