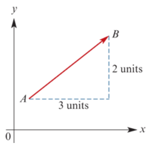
Vectors

- **vector:** a directed line segment

- arrow indicates direction
- length indicates magnitude
- notated as $\vec{a}, \widetilde{A}, \underset{\sim}{a}$
- column notation: $\begin{bmatrix} x \\ y \end{bmatrix}$
- vectors with equal magnitude and direction are equivalent



Vector addition

$\boldsymbol{u} + \boldsymbol{v}$ can be represented by drawing each vector head to tail then joining the lines.
Addition is commutative (parallelogram)

Scalar multiplication

For $k \in \mathbb{R}^+$, $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$ but length is multiplied by a factor of k.

When multiplied by $k < 0$, direction is reversed and length is multplied by k.

Vector subtraction

To find $\boldsymbol{u} - \boldsymbol{v}$, add $-\boldsymbol{v}$ to $\boldsymbol{u}$

Parallel vectors

Parallel vectors have same direction or opposite direction.

Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel if there is some $k \in \mathbb{R} \setminus \{0\}$ such at $\boldsymbol{u} = k\boldsymbol{v}$

Position vectors

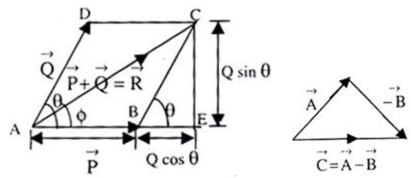
Vectors may describe a position relative to O.

For a point A, the position vector is $\overrightarrow{OA}$

Linear combinations of non-parallel vectors

If two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel, then:

$$m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b} \quad \therefore \quad m = p,\ n = q$$



Column vector notation

A vector between points $A(x_1, y_1)$, $B(x_2, y_2)$ can be represented as $\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$

Component notation

A vector $\boldsymbol{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ can be written as $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$.
$\boldsymbol{u}$ is the sum of two components $x\boldsymbol{i}$ and $y\boldsymbol{j}$
Magnitude of vector $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$ is denoted by $|u| = \sqrt{x^2 + y^2}$

Basic algebra applies:
$(x\boldsymbol{i} + y\boldsymbol{j}) + (m\boldsymbol{i} + n\boldsymbol{j}) = (x + m)\boldsymbol{i} + (y + n)\boldsymbol{j}$
Two vectors equal if and only if their components are equal.

Unit vectors

A vector of length 1. $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors.

A unit vector in direction of $\boldsymbol{a}$ is denoted by $\hat{\boldsymbol{a}}$:

$$\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a} \quad (\implies |\hat{\boldsymbol{a}}| = 1)$$

Also, unit vector of $\boldsymbol{a}$ can be defined by $\boldsymbol{a} \cdot |\boldsymbol{a}|$

Scalar products / dot products

If $\boldsymbol{a} = a_i\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_i\boldsymbol{i} + b_2\boldsymbol{j}$, the dot product is:

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$

Produces a real number, not a vector.

$$\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$$

Scalar product properties

1. $k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (kb)$
2. $\boldsymbol{a} \cdot \boldsymbol{0} = 0$
3. $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$

For parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{cases} |\boldsymbol{a}||\boldsymbol{b}| & \text{if same direction} \\ -|\boldsymbol{a}||\boldsymbol{b}| & \text{if opposite directions} \end{cases}$$

Geometric scalar products

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$$

where $0 \leq \theta \leq \pi$

Perpendicular vectors

If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, then $\boldsymbol{a} \perp \boldsymbol{b}$ (since $\cos 90 = 0$)

Finding angle between vectors

$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{a_1 b_1 + a_2 b_2}{|\boldsymbol{a}||\boldsymbol{b}|}$$

Vector projections

Vector resolute of $\boldsymbol{a}$ in direction of $\boldsymbol{b}$ is magnitude of $\boldsymbol{a}$ in direction of $\boldsymbol{b}$.

$$\boldsymbol{u} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|^2}\boldsymbol{b} = \left(\boldsymbol{a} \cdot \frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right)\left(\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right) = (\boldsymbol{a} \cdot \hat{\boldsymbol{b}})\hat{\boldsymbol{b}}$$

Vector proofs

Concurrent lines - ≥ 3 lines intersect at a single point
Collinear points - ≥ 3 points lie on the same line

Useful vector properties:

- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then $\boldsymbol{b} = k\boldsymbol{a}$ for some $k \in \mathbb{R} \setminus \{0\}$
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel with at least one point in common, then they lie on the same straight line
- Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$