Vectors

- **vector:** a directed line segment

- arrow indicates direction
- length indicates magnitude
- notated as $\vec{a}, \widetilde{A}, \mathbf{a}$
- column notation: $\begin{bmatrix} x \\ y \end{bmatrix}$
- vectors with equal magnitude and direction are equivalent

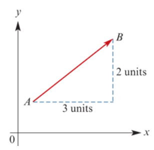


Figure 1:

Vector addition

$u + v$ can be represented by drawing each vector head to tail then joining the lines.
Addition is commutative (parallelogram)

Scalar multiplication

For $k \in \mathbb{R}^+$, ku has the same direction as u but length is multiplied by a factor of k.

When multiplied by $k < 0$, direction is reversed and length is multplied by k.

Vector subtraction

To find $u - v$, add $-v$ to u

Parallel vectors

Parallel vectors have same direction or opposite direction.

Two non-zero vectors u and v are parallel if there is some $k \in \mathbb{R} \ \{0\}$ such at $u = kv$

Position vectors

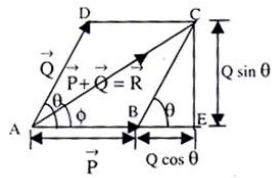
Vectors may describe a position relative to O.

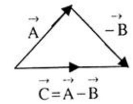
For a point A, the position vector is $\overrightarrow{OA}$

Linear combinations of non-parallel vectors

If two non-zero vectors a and b are not parallel, then:

$$ma + nb = pa + qb \quad \therefore \quad m = p, \ n = q$$





Column vector notation

A vector between points $A(x_1, y_1)$, $B(x_2, y_2)$ can be represented as $\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$

Component notation

A vector $u = \begin{bmatrix} x \\ y \end{bmatrix}$ can be written as $u = xi + yj$.
u is the sum of two components xi and yj
Magnitude of vector $u = xi + yj$ is denoted by $|u| = \sqrt{x^2 + y^2}$

Basic algebra applies:
$(xi + yj) + (mi + nj) = (x + m)i + (y + n)j$
Two vectors equal if and only if their components are equal.

Unit vectors

A vector of length 1. i and j are unit vectors.

A unit vector in direction of a is denoted by $\hat{a}$:

$$\hat{a} = \frac{1}{|a|} a \quad (\implies |\hat{a}| = 1)$$

Also, unit vector of a can be defined by $a \cdot |a|$

Scalar products / dot products

If $a = a_i i + a_2 j$ and $b = b_i i + b_2 j$, the dot product is:

$$a \cdot b = a_1 b_1 + a_2 b_2$$

Produces a real number, not a vector.

$$a \cdot a = |a|^2$$

on CAS: `dotP([a b c], [d e f])`

Scalar product properties

1. $k(a \cdot b) = (ka) \cdot b = a \cdot (kb)$
2. $a \cdot 0 = 0$
3. $a \cdot (b + c) = a \cdot b + a \cdot c$
4. $i \cdot i = j \cdot j = k \cdot k = 1$
5. If $a \cdot b = 0$, a and b are perpendicular
6. $a \cdot a = |a|^2 = a^2$

For parallel vectors a and b:

$$a \cdot b = \begin{cases} |a||b| & \text{if same direction} \\ -|a||b| & \text{if opposite directions} \end{cases}$$

Geometric scalar products

$$a \cdot b = |a||b| \cos \theta$$

where $0 \leq \theta \leq \pi$

Perpendicular vectors

If $a \cdot b = 0$, then $a \perp b$ (since $\cos 90 = 0$)

Finding angle between vectors

positive direction

$$\cos \theta = \frac{a \cdot b}{|a||b|} = \frac{a_1 b_1 + a_2 b_2}{|a||b|}$$

on CAS: `angle([a b c], [a b c])` (Action -> Vector -> Angle)

Vector projections

Vector resolute of a in direction of b is magnitude of a in direction of b.

$$u = \frac{a \cdot b}{|b|^2} b = \left(a \cdot \frac{b}{|b|} \right) \left(\frac{b}{|b|} \right) = (a \cdot \hat{b}) \hat{b}$$

Scalar resolute of $\vec{a}$ on $\vec{b} = |\vec{u}| = \vec{a} \cdot \hat{\vec{b}}$ (results in a scalar)
Vector resolute of $\vec{a}$ perpendicular to b is equal to $\vec{a} - \vec{u}$ where $\vec{u}$ is vector projection of $\vec{a}$ on $\vec{b}$

Vector proofs

Concurrent lines - ≥ 3 lines intersect at a single point
Collinear points - ≥ 3 points lie on the same line ($\implies \vec{OC} = \lambda \vec{OA} + \mu \vec{OB}$ where $\lambda + \mu = 1$. If C is between $\vec{AB}$, then $0 < \mu < 1$)

Useful vector properties:

- If a and b are parallel, then $b = ka$ for some $k \in \mathbb{R} \ \{0\}$
- If a and b are parallel with at least one point in common, then they lie on the same straight line
- Two vectors a and b are perpendicular if $a \cdot b = 0$
- $a \cdot a = |a|^2$

Linear dependence

Vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent if they are non-parallel and:

$$k\vec{a} + l\vec{b} + m\vec{c} = 0$$
$$\therefore \vec{c} = m\vec{a} + n\vec{b} \quad \text{(simultaneous)}$$

$\vec{a}, \vec{b}$, and $\vec{c}$ are linearly independent if no vector in the set is expressible as a linear combination of other vectors in set, or if they are parallel.

Vector $\vec{w}$ is a linear combination of vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$

Three-dimensional vectors

Right-hand rule for axes - z is up or out of page.

Angle between vector and axis

Direction of a vector can be given by the angles it makes with $\vec{i}, \vec{j}, \vec{k}$ directions.

For $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$ which makes angles α, β, γ with positive direction of x, y, z axes:

$$\cos \alpha = \frac{a_1}{|\vec{a}|}, \quad \cos \beta = \frac{a_2}{|\vec{a}|}, \quad \cos \gamma = \frac{a_3}{|\vec{a}|}$$

on CAS: `angle([a b c], [1 0 0])` for angle between $a\vec{i} + b\vec{j} + c\vec{k}$ and x-axis

Collinearity

Points A, B, C are collinear iff $\vec{AC} = m\vec{AB}$ where $m \neq 0$