1 Statistics

Continuous random variables

A continuous random variable X has a pdf f such that:

1. $f(x) \geq 0 \forall x$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$E(X) = \int_{\mathbf{X}} (x \cdot f(x))\,dx$$
$$\operatorname{Var}(X) = E\left[(X-\mu)^2\right]$$

$$\Pr(X \leq c) = \int_{-\infty}^{c} f(x)\,dx$$

Two random variables X, Y

If X and Y are independent:

$$\mathrm{E}(aX + bY) = a\,\mathrm{E}(X) + b\,\mathrm{E}(Y)$$
$$\operatorname{Var}(aX \pm bY \pm c) = a^2 \operatorname{Var}(X) + b^2 \operatorname{Var}(Y)$$

Linear functions $X \to aX + b$

$$\begin{aligned}\Pr(Y \leq y) &= \Pr(aX + b \leq y) \\ &= \Pr\left(X \leq \frac{y-b}{a}\right) \\ &= \int_{-\infty}^{\frac{y-b}{a}} f(x)\,dx\end{aligned}$$

Mean: $\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b$

Variance: $\operatorname{Var}(aX + b) = a^2 \operatorname{Var}(X)$

Expectation theorems

For some non-linear function g, the expected value $E(g(X))$ is not equal to $g(E(X))$.

$$E(X^2) = \text{Var}(X) - [E(X)]^2$$

$$E(X^n) = \Sigma x^n \cdot p(x) \qquad \text{(non-linear)}$$

$$\neq [E(X)]^n$$

$$E(aX \pm b) = aE(X) \pm b \qquad \text{(linear)}$$

$$E(b) = b \qquad (\forall b \in \mathbb{R})$$

$$E(X + Y) = E(X) + E(Y) \qquad \text{(two variables)}$$

Sample mean

Approximation of the **population mean** determined experimentally.

$$\overline{x} = \frac{\Sigma x}{n}$$

where

n is the size of the sample (number of sample points)

x is the value of a sample point

On CAS

1. Spreadsheet
2. In cell A1:
   `mean(randNorm(sd, mean, sample size))`
3. Edit → Fill → Fill Range
4. Input range as A1:An where n is the number of samples
5. Graph → Histogram

Sample size of n

$$\overline{X} = \sum_{i=1}^{n} \frac{x_i}{n} = \frac{\sum x}{n}$$

Sample mean is distributed with mean μ and sd $\frac{\sigma}{\sqrt{n}}$ (approaches these values for increasing sample size n).

For a new distribution with mean of n trials, $\text{E}(X') = \text{E}(X), \quad \text{sd}(X') = \dfrac{\text{sd}(X)}{\sqrt{n}}$

On CAS

- Spreadsheet → Catalog → `randNorm(sd, mean, n)` where `n` is the number of samples. Show histogram with Histogram key in top left
- To calculate parameters of a dataset: Calc → One-variable

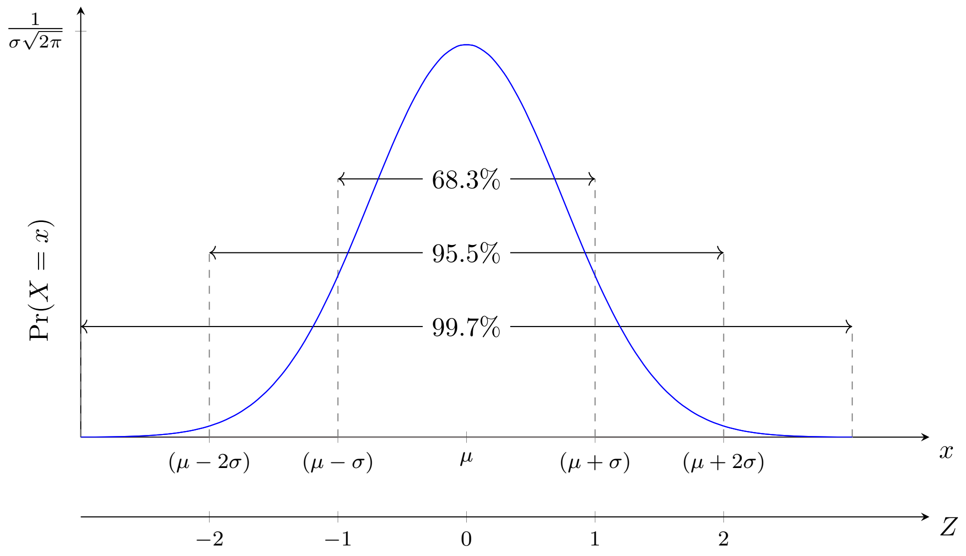
Normal distributions

$$Z = \frac{X - \mu}{\sigma}$$

Normal distributions must have area (total prob.) of 1 $\implies \int_{-\infty}^{\infty} f(x)\,dx = 1$

mean = mode = median

Always express z as +ve. Express confidence interval as ordered pair.



Central limit theorem

If X is randomly distributed with mean μ and sd σ, then with an adequate sample size n the distribution of the sample mean $\overline{X}$ is approximately normal with mean $E(\overline{X})$ and $\text{sd}(\overline{X}) = \frac{\sigma}{\sqrt{n}}$.

Confidence intervals

- **Point estimate:** single-valued estimate of the population mean from the value of the sample mean $\overline{x}$
- **Interval estimate:** confidence interval for population mean μ
- $C\%$ confidence interval $\implies$ $C\%$ of samples will contain population mean μ

95% confidence interval

For 95% c.i. of population mean μ:

$$x \in \left(\overline{x} \pm 1.96\frac{\sigma}{\sqrt{n}}\right)$$

where:

$\overline{x}$ is the sample mean

σ is the population sd

n is the sample size from which $\overline{x}$ was calculated

On CAS

Menu → Stats → Calc → Interval

Set *Type = One-Sample Z Int*

and select *Variable*

Margin of error

For 95% confidence interval of μ:

$$M = 1.96 \times \frac{\sigma}{\sqrt{n}}$$
$$\implies n = \left(\frac{1.96\sigma}{M}\right)^2$$

Always round n up to a whole number of samples.

General case

For C% c.i. of population mean μ:

$$x \in \left(\overline{x} \pm k\frac{\sigma}{\sqrt{n}}\right)$$

where k is such that $\Pr(-k < Z < k) = \frac{C}{100}$

Confidence interval for multiple trials

For a set of n confidence intervals (samples), there is 0.95^n chance that all n intervals contain the population mean μ.

2 Hypothesis testing

Note hypotheses are always expressed in terms of population parameters

Null hypothesis H_0

Sample drawn from population has same mean as control population, and any difference can be explained by sample variations.

Alternative hypothesis H_1

Amount of variation from control is significant, despite standard sample variations.

p-value

$$p = \Pr(\overline{X} \lessgtr \mu(H_1))$$
$$= 2 \cdot \Pr(\overline{X} <> \mu(H_1) | \mu = 8)$$

Probability of observing a value of the sample statistic as significant as the one observed, assuming null hypothesis is true.

p	Conclusion
> 0.05	insufficient evidence against H_0
< 0.05 (5%)	good evidence against H_0
< 0.01 (1%)	strong evidence against H_0
< 0.001 (0.1%)	very strong evidence against H_0

Statistical significance

Significance level is denoted by α.

If $p < \alpha$, null hypothesis is **rejected**

If $p > \alpha$, null hypothesis is **accepted**

z-test

Hypothesis test for a mean of a sample drawn from a normally distributed population with a known standard deviation.

On CAS

Menu → Statistics → Calc → Test.

Select *One-Sample Z-Test* and *Variable*, then input:

μ cond: same operator as H_1

μ_0: expected sample mean (null hypothesis)

σ: standard deviation (null hypothesis)

$\overline{x}$: sample mean

n: sample size

One-tail and two-tail tests

One tail

- μ has changed in one direction
- State "$H_1 : \mu \lesseqgtr$ known population mean"

Two tail

- Direction of $\Delta\mu$ is ambiguous
- State "$H_1 : \mu \neq$ known population mean"

For two tail tests:

$$p\text{-value} = \Pr(|\overline{X} - \mu| \geq |\overline{x}_0 - \mu|)$$
$$= \left(|Z| \geq \left|\frac{\overline{x}_0 - \mu}{\sigma \div \sqrt{n}}\right|\right)$$

Modulus notation for two tail

$\Pr(|\overline{X} - \mu| \geq a) \implies$ "the probability that the distance between $\overline{\mu}$ and μ is $\geq a$"

Inverse normal

On CAS

invNormCdf("L", α, $\dfrac{\sigma}{n^{\alpha}}$, μ)

Errors

Type I error H_0 is rejected when it is **true**

Type II error H_0 is **not** rejected when it is **false**