

1 Probability

Probability theorems

Union:	$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
Multiplication theorem:	$\Pr(A \cap B) = \Pr(A B) \times \Pr(B)$
Conditional:	$\Pr(A B) = \frac{\Pr(A \cap B)}{\Pr(B)}$
Law of total probability:	$\Pr(A) = \Pr(A B) \cdot \Pr(B) + \Pr(A B') \cdot \Pr(B')$

Mutually exclusive $\implies \Pr(A \cup B) = 0$

Independent events:

$$\begin{aligned}\Pr(A \cap B) &= \Pr(A) \times \Pr(B) \\ \Pr(A|B) &= \Pr(A) \\ \Pr(B|A) &= \Pr(B)\end{aligned}$$

Discrete random distributions

Any experiment or activity involving chance will have a probability associated with each result or *outcome*. If the outcomes have a reference to **discrete numeric values** (outcomes that can be counted), and the result is unknown, then the activity is a *discrete random probability distribution*.

Discrete probability distributions

If an activity has outcomes whose probability values are all positive and less than one ($\implies 0 \leq p(x) \leq 1$), and for which the sum of all outcome probabilities is unity ($\implies \sum p(x) = 1$), then it is called a *probability distribution* or *probability mass function*.

- **Probability distribution graph** - a series of points on a cartesian axis representing results of outcomes. $\Pr(X = x)$ is on y -axis, x is on x axis.
- **Mean μ or expected value $E(X)$** - measure of central tendency. Also known as *balance point*. Centre of a symmetrical distribution.

$$\begin{aligned}\bar{x} = \mu = E(X) &= \frac{\Sigma [x \cdot f(x)]}{\Sigma f} && \text{(where } f = \text{absolute frequency)} \\ &= \sum_{i=1}^n [x_i \cdot \Pr(X = x_i)] && \text{(for } n \text{ values of } x) \\ &= \int_{-\infty}^{\infty} (x \cdot f(x)) dx && \text{(for pdf } f)\end{aligned}$$

- **Mode** - most popular value (has highest probability of X values). Multiple modes can exist if > 1 X value have equal-highest probability. Number must exist in distribution.
- **Median m** - the value of x such that $\Pr(X \leq m) = \Pr(X \geq m) = 0.5$. If $m > 0.5$, then value of X that is reached is the median of X . If $m = 0.5 = 0.5$, then m is halfway between this value and the next. To find m , add values of X from smallest to largest until the sum reaches 0.5.

$$m = X \text{ such that } \int_{-\infty}^m f(x) dx = 0.5$$

- **Variance σ^2** - measure of spread of data around the mean. Not the same magnitude as the original data.

For distribution $x_1 \mapsto p_1, x_2 \mapsto p_2, \dots, x_n \mapsto p_n$:

$$\begin{aligned}\sigma^2 &= \text{Var}(x) = \sum_{i=1}^n p_i(x_i - \mu)^2 \\ &= \sum (x - \mu)^2 \times \Pr(X = x) \\ &= \sum x^2 \times p(x) - \mu^2 \\ &= E(X^2) - [E(X)]^2\end{aligned}$$

- **Standard deviation** σ - measure of spread in the original magnitude of the data. Found by taking square root of the variance:

$$\begin{aligned}\sigma &= \text{sd}(X) \\ &= \sqrt{\text{Var}(X)}\end{aligned}$$

Expectation theorems

For some non-linear function g , the expected value $E(g(X))$ is not equal to $g(E(X))$.

$$\begin{aligned}E(X^n) &= \sum x^n \cdot p(x) && \text{(non-linear function)} \\ &\neq [E(X)]^n \\ E(aX \pm b) &= aE(X) \pm b && \text{(linear function)} \\ E(b) &= b && \text{(for constant } b \in \mathbb{R}) \\ E(X + Y) &= E(X) + E(Y) && \text{(for two random variables)}\end{aligned}$$

Variance theorems

$$\text{Var}(aX \pm bY \pm c) = a^2 \text{Var}(X) + b^2 \text{Var}(Y)$$

2 Binomial Theorem

$$\begin{aligned}(x + y)^n &= \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n \\ &= \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k \\ &= \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}\end{aligned}$$

Patterns

1. powers of x decrease $n \rightarrow 0$
2. powers of y increase $0 \rightarrow n$
3. coefficients are given by n th row of Pascal's Triangle where $n = 0$ has one term
4. Number of terms in $(x + a)^n$ expanded & simplified is $n + 1$

Combinatorics

$$\text{Binomial coefficient: } {}^n C_r = \binom{n}{k}$$

- Arrangements $\binom{n}{k} = \frac{n!}{(n-r)!}$
- Combinations $\binom{n}{k} = \frac{n!}{r!(n-r)!}$
- Note $\binom{n}{k} = \binom{n}{n-k}$

On CAS: (soft keyboard) $\boxed{\downarrow}$ \rightarrow $\boxed{\text{Advanced}}$ \rightarrow $\text{nCr}(n, cr)$

Pascal's Triangle

$n =$																
0								1								
1							1	1								
2							1	2	1							
3							1	3	3	1						
4								1	4	6	4	1				
5									1	5	10	10	5	1		
6										1	6	15	20	15	6	1

3 Binomial distributions

(aka Bernoulli distributions)

$$\begin{aligned} \text{Defined by } X &\sim \text{Bi}(n, p) \\ \implies \Pr(X = x) &= \binom{n}{x} p^x (1-p)^{n-x} \\ &= \binom{n}{x} p^x q^{n-x} \end{aligned}$$

where:

 n is the number of trialsThere are two possible outcomes: S or F $\Pr(\text{success}) = p$ $\Pr(\text{failure}) = 1 - p = q$ **Conditions for a binomial variable/distribution**

1. Two possible outcomes: **success** or **failure**
2. $\Pr(\text{success})$ is constant across trials (also denoted p)
3. Finite number n of independent trials

Solve on CASMain \rightarrow Interactive \rightarrow Distribution \rightarrow `binomialPdf`Input `x` (no. of successes), `numtrial` (no. of trials), `pos` (probability of success)**Properties of $X \sim \text{Bi}(n, p)$**

$$\begin{aligned} \text{Mean} & \quad \mu(X) = np \\ \text{Variance} & \quad \sigma^2(X) = np(1-p) \\ \text{s.d.} & \quad \sigma(X) = \sqrt{np(1-p)} \end{aligned}$$

Applications of binomial distributions

$$\Pr(X \geq a) = 1 - \Pr(X < a)$$

4 Continuous probability**Continuous random variables**

- a variable that can take any real value in an interval

Probability density functions

- area under curve = 1 $\implies \int f(x) dx = 1$
- $f(x) \geq 0 \forall x$
- pdfs may be linear
- must show sections where $f(x) = 0$ (use open/closed circles)

$$Pr(a \leq X \leq b) = \int_a^b f(x) dx$$