Year 12 Methods

Andrew Lorimer

1   Functions

- vertical line test
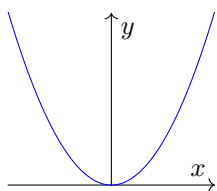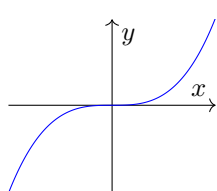- each x value produces only one y value

One to one functions

- $f(x)$ is *one to one* if $f(a) \neq f(b)$ if $a, b \in \text{dom}(f)$ and $a \neq b$

  $\implies$ unique y for each x ($\sin x$ is not 1:1, x^3 is)
- horizontal line test
- if not one to one, it is many to one

Odd and even functions

$$\text{Even:} \qquad f(x) = f(-x)$$

$$\text{Odd:} \qquad -f(x) = f(-x)$$

Even $\implies$ symmetrical across y-axis

$x^{\pm \frac{p}{q}}$ is odd if q is odd

For x^n, parity of $n \equiv$ parity of function

Even: **Odd:**

Inverse functions

- Inverse of $f(x)$ is denoted $f^{-1}(x)$
- f must be one to one
- If $f(g(x)) = x$, then g is the inverse of f
- Represents reflection across $y = x$
- $\implies f^{-1}(x) = f(x)$ intersections lie on $y = x$
- ran $f = \text{dom } f^{-1}$

  dom $f = \text{ran } f^{-1}$
- "Inverse" $\neq$ "inverse *function*" (functions must pass vertical line test)

Finding f^{-1}

1. Let $y = f(x)$
2. Swap x and y ("take inverse"
3. Solve for y

   Sqrt: state $\pm$ solutions then restrict
4. State rule as $f^{-1}(x) = \ldots$
5. For inverse *function*, state in function notation

Simultaneous equations (linear)

- **Unique solution** - lines intersect at point
- **Infinitely many solutions** - lines are equal
- **No solution** - lines are parallel

Solving $\begin{cases} px + qy = a \\ rx + sy = b \end{cases}$ **for $\{0, 1, \infty\}$ solutions**

where all coefficients are known except for one, and a, b are known

1. Write as matrices: $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$
2. Find determinant of first matrix: $\Delta = ps - qr$
3. Let $\Delta = 0$ for number of solutions $\neq 1$

   or let $\Delta \neq 0$ for one unique solution.
4. Solve determinant equation to find variable

   For infinite/no solutions:
5. Substitute variable into both original equations
6. Rearrange equations so that LHS of each is the same
7. $\text{RHS}(1) = \text{RHS}(2) \implies (1) = (2) \, \forall x \, (\infty \text{ solns})$

   $\text{RHS}(1) \neq \text{RHS}(2) \implies (1) \neq (2) \, \forall x \, (0 \text{ solns})$

On CAS: Matrix $\rightarrow$ `det`

Solving $\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3 \end{cases}$

- Use elimination

- Generate two new equations with only two variables
- Rearrange & solve
- Substitute one variable into another equation to find another variable

Piecewise functions

$$\text{e.g.} \quad f(x) = \begin{cases} x^{1/3}, & x \leq 0 \\ 2, & 0 < x < 2 \\ x, & x \geq 2 \end{cases}$$

Open circle: point included

Closed circle: point not included

Operations on functions

For $f \pm g$ and $f \times g$: $\text{dom}' = \text{dom}(f) \cap \text{dom}(g)$

Addition of linear piecewise graphs: add y-values at key points

Product functions:

- product will equal 0 if $f = 0$ or $g = 0$
- $f'(x) = 0 \veebar g'(x) = 0 \not\Rightarrow (f \times g)'(x) = 0$

Composite functions

$(f \circ g)(x)$ is defined iff $\text{ran}(g) \subseteq \text{dom}(f)$

2   Polynomials

Linear equations

Forms

- $y = mx + c$
- $\frac{x}{a} + \frac{y}{b} = 1$ where (x_1, y_1) lies on the graph
- $y - y_1 = m(x - x_1)$ where $(a, 0)$ and $(0, b)$ are x- and y-intercepts

Line properties

Parallel lines: $m_1 = m_2$

Perpendicular lines: $m_1 \times m_2 = -1$

Distance: $|\vec{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Quadratics

$$x^2 + bx + c = (x + m)(x + n)$$

$$\text{where } mn = c, \ m + n = b$$

Difference of squares

$$a^2 - b^2 = (a - b)(a + b)$$

Perfect squares

$$a^2 \pm 2ab + b^2 = (a \pm b^2)$$

Completing the square

$$x^2 + bx + c = (x + \frac{b}{2})^2 + c - \frac{b^2}{4}$$

$$ax^2 + bx + c = a(x - \frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{(Discriminant } \Delta = b^2 - 4ac)$$

Cubics

Difference of cubes

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Sum of cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Perfect cubes

$$a^3 \pm 3a^2b + 3ab^2 \pm b^3 = (a \pm b)^3$$

$$y = a(bx - h)^3 + c$$

- $m = 0$ at *stationary point of inflection* (i.e. $(\frac{h}{b}, k)$)
- $y = (x - a)^2(x - b)$ — max at $x = a$, min at $x = b$
- $y = a(x - b)(x - c)(x - d)$ — roots at b, c, d
- $y = a(x - b)^2(x - c)$ — roots at b (instantaneous), c (intercept)

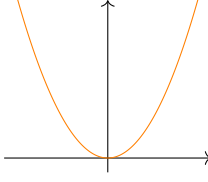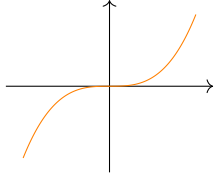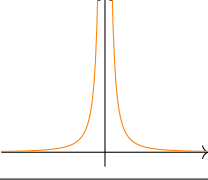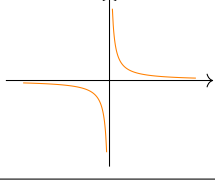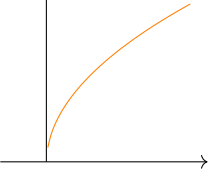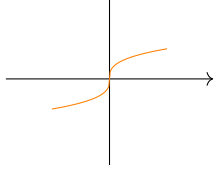## Quartic graphs

Forms of quartic equations

$y = ax^4$

$y = a(x - b)(x - c)(x - d)(x - e)$

$y = ax^4 + cd^2 (c \geq 0)$

$y = ax^2(x - b)(x - c)$

$y = a(x - b)^2(x - c)^2$

$y = a(x - b)(x - c)^3$

	n is even	n is odd
$x^n, n \in \mathbb{Z}^+$		
$x^n, n \in \mathbb{Z}^-$		
$x^{\frac{1}{n}}, n \in \mathbb{Z}^-$		

3   Transformations

Order of operations: DRT

$$\text{dilations — reflections — translations}$$

Transforming x^n to $a(x-h)^n + K$

- dilation factor of $|a|$ units parallel to y-axis or from x-axis
- if $a < 0$, graph is reflected over x-axis
- translation of k units parallel to y-axis or from x-axis
- translation of h units parallel to x-axis or from y-axis
- for $(ax)^n$, dilation factor is $\frac{1}{a}$ parallel to x-axis or from y-axis
- when $0 < |a| < 1$, graph becomes closer to axis

Transforming $f(x)$ to $y = Af[n(x+c)] + b$

Applies to exponential, log, trig, e^x, polynomials.

Functions must be written in form $y = Af[n(x+c)]+b$

- dilation by factor $|A|$ from x-axis (if $A < 0$, reflection across y-axis)
- dilation by factor $\frac{1}{n}$ from y-axis (if $n < 0$, reflection across x-axis)
- translation of c units from y-axis (x-shift)

- translation of b units from x-axis (y-shift)

Dilations

Two pairs of equivalent processes for $y = f(x)$:

1. - Dilating from x-axis: $(x, y) \rightarrow (x, by)$
   - Replacing y with $\frac{y}{b}$ to obtain $y = bf(x)$

2. - Dilating from y-axis: $(x, y) \rightarrow (ax, y)$
   - Replacing x with $\frac{x}{a}$ to obtain $y = f(\frac{x}{a})$

For graph of $y = \frac{1}{x}$, horizontal & vertical dilations are equivalent (symmetrical). If $y = \frac{a}{x}$, graph is contracted rather than dilated.

Matrix transformations

Find new point (x', y'). Substitute these into original equation to find image with original variables (x, y).

Reflections

- Reflection **in** axis = reflection **over** axis = reflection **across** axis
- Translations do not change

Translations

For $y = f(x)$, these processes are equivalent:

- applying the translation $(x, y) \to (x + h, y + k)$ to the graph of $y = f(x)$

- replacing x with $x - h$ and y with $y - k$ to obtain $y - k = f(x - h)$

Power functions

Mostly only on CAS.

We can write $x^{\frac{-1}{n}} = \frac{1}{x^{\frac{1}{n}}} = \frac{1}{\sqrt[n]{x}}$n.

Domain is: $\begin{cases} \mathbb{R} \setminus \{0\} & \text{if } n \text{ is odd} \\ \mathbb{R}^+ & \text{if } n \text{ is even} \end{cases}$

If n is odd, it is an odd function.

$x^{\frac{p}{q}}$ **where** $p, q \in \mathbb{Z}^+$

$$x^{\frac{p}{q}} = \sqrt[q]{x^p}$$

- if $p > q$, the shape of x^p is dominant
- if $p < q$, the shape of $x^{\frac{1}{q}}$ is dominant
- points $(0, 0)$ and $(1, 1)$ will always lie on graph
- Domain is: $\begin{cases} \mathbb{R} & \text{if } q \text{ is odd} \\ \mathbb{R}^+ \cup \{0\} & \text{if } q \text{ is even} \end{cases}$

4   Exponentials & Logarithms

Logarithmic identities

$$\log_b(xy) = \log_b x + \log_b y$$
$$\log_b x^n = n \log_b x$$
$$\log_b y^{x^n} = x^n \log_b y$$
$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$
$$\log_a(m^{-1}) = -\log_a m$$
$$\log_b c = \frac{\log_a c}{\log_a b}$$

Index identities

$$b^{m+n} = b^m \cdot b^n$$
$$(b^m)^n = b^{m \cdot n}$$
$$(b \cdot c)^n = b^n \cdot c^n$$
$$b^m \div a^n = b^{m-n}$$

Inverse functions

For $f : \mathbb{R} \to \mathbb{R}, f(x) = a^x$, inverse is:

$$f^{-1} : \mathbb{R}^+ \to \mathbb{R}, f^{-1} = \log_a x$$

Euler's number e

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$
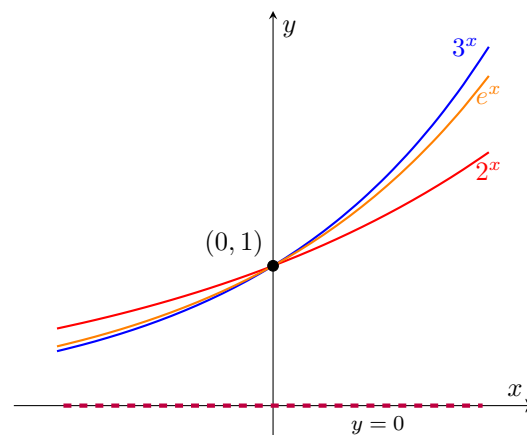
Modelling

$$A = A_0 e^{kt}$$

- A_0 is initial value
- t is time taken
- k is a constant
- For continuous growth, $k > 0$
- For continuous decay, $k < 0$

Graphing exponential functions

$$f(x) = Aa^{k(x-b)} + c, \quad | \; a > 1$$

- **y-intercept** at $(0, A \cdot a^{-kb} + c)$ as $x \to \infty$
- **horizontal asymptote** at $y = c$
- **domain** is $\mathbb{R}$
- **range** is (c, ∞)
- dilation of factor $|A|$ from x-axis
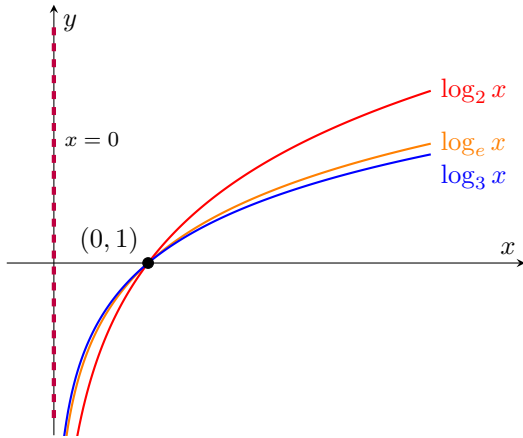- dilation of factor $\frac{1}{k}$ from y-axis

Graphing logarithmic functions

$\log_e x$ is the inverse of e^x (reflection across $y = x$)

$$f(x) = A \log_a k(x - b) + c$$

where

- **domain** is (b, ∞)
- **range** is $\mathbb{R}$
- **vertical asymptote** at $x = b$
- y-intercept exists if $b < 0$
- dilation of factor $|A|$ from x-axis
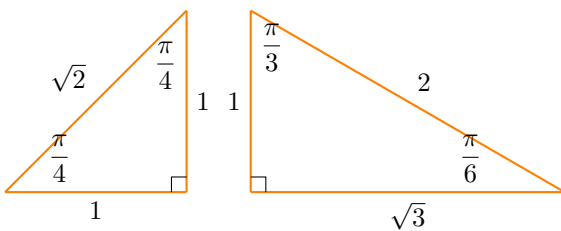- dilation of factor $\frac{1}{k}$ from y-axis

Finding equations

On CAS: $\begin{cases} \texttt{f(3)=9} \\ \texttt{g(3)=0} \end{cases} \Big|_{\texttt{a,b}}$

5   Circular functions

Radians and degrees

$$1 \text{ rad} = \frac{180 \deg}{\pi}$$

Exact values

Compound angle formulas

$$\cos(x \pm y) = \cos x + \cos y \mp \sin x \sin y$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

Double angle formulas

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 1 - 2\sin^2 x$$
$$= 2\cos^2 x - 1$$
$$\sin 2x = 2 \sin x \cos x$$
$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Symmetry

$$\sin(\theta + \frac{\pi}{2}) = \sin \theta$$
$$\sin(\theta + \pi) = -\sin \theta$$

$$\cos(\theta + \frac{\pi}{2}) = -\cos \theta$$
$$\cos(\theta + \pi) = -\cos(\theta + \frac{3\pi}{2})$$
$$= \cos(-\theta)$$

Complementary relationships

$$\sin \theta = \cos(\frac{\pi}{2} - \theta)$$
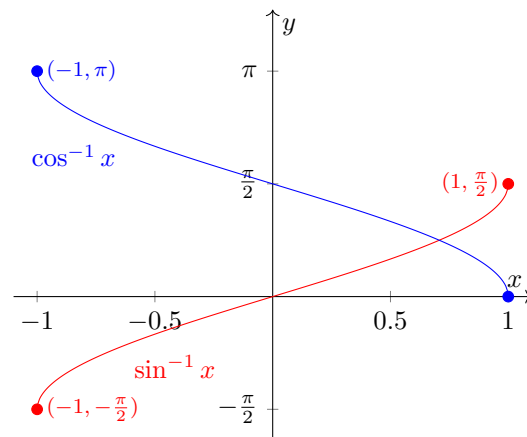$$= -\cos(\theta + \frac{\pi}{2})$$
$$\cos \theta = \sin(\frac{\pi}{2} - \theta)$$
$$= \sin(\theta + \frac{\pi}{2})$$

Pythagorean identity

$$\cos^2 \theta + \sin^2 \theta = 1$$

Inverse circular functions

Inverse functions: $f(f^{-1}(x)) = x$ (restrict domain)

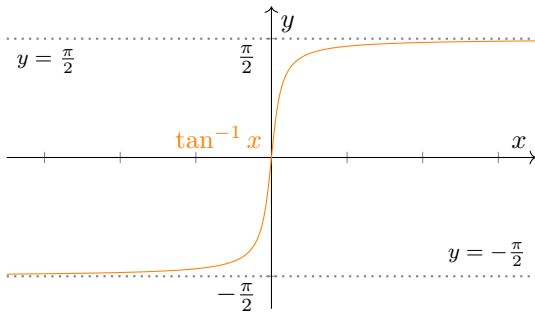$$\sin^{-1} : [-1, 1] \to \mathbb{R}, \quad \sin^{-1} x = y$$

$$\text{where } \sin y = x, \; y \in \left[\tfrac{-\pi}{2}, \tfrac{\pi}{2}\right]$$

$$\cos^{-1} : [-1, 1] \to \mathbb{R}, \quad \cos^{-1} x = y$$

$$\text{where } \cos y = x, \; y \in [0, \pi]$$

$$\tan^{-1} : \mathbb{R} \to \mathbb{R}, \quad \tan^{-1} x = y$$

$$\text{where } \tan y = x, \; y \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

sin and cos graphs

$$f(x) = a\sin(bx - c) + d$$

where:

Period $= \frac{2\pi}{n}$

dom $= \mathbb{R}$

ran $= [-b + c, b + c]$;

$\cos(x)$ starts at $(0, 1)$, $\sin(x)$ starts at $(0, 0)$

0 amplitidue $\implies$ straight line

$a < 0$ or $b < 0$ inverts phase (swap sin and cos)

$c = T = \frac{2\pi}{b} \implies$ no net phase shift

tan graphs

$$y = a\tan(nx)$$

Period $= \dfrac{\pi}{n}$

Range is $\mathbb{R}$

Roots at $x = \dfrac{k\pi}{n}$ where $k \in \mathbb{Z}$

Asymptotes at $x = \frac{(2k+1)\pi}{2n}$

Asymptotes should always have equations

Solving trig equations

1. Solve domain for $n\theta$
2. Find solutions for $n\theta$
3. Divide solutions by n

$\sin 2\theta = \frac{\sqrt{3}}{2}, \quad \theta \in [0, 2\pi] \quad (\therefore 2\theta \in [0, 4\pi])$

$2\theta = \sin^{-1}\frac{\sqrt{3}}{2}$

$2\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$

$\therefore \theta = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$

6   Calculus

Average rate of change

$$m \text{ of } x \in [a, b] = \frac{f(b) - f(a)}{b - a} = \frac{dy}{dx}$$

On CAS: Action $\to$ Calculation $\to$ `diff`

Average value

$$f_{\text{avg}} = \frac{1}{b - a} \int_a^b f(x)\, dx$$

Instantaneous rate of change

Secant - line passing through two points on a curve

Chord - line segment joining two points on a curve

Limit theorems

1. For constant function $f(x) = k$, $\lim_{x \to a} f(x) = k$
2. $\lim_{x \to a}(f(x) \pm g(x)) = F \pm G$
3. $\lim_{x \to a}(f(x) \times g(x)) = F \times G$
4. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G}, G \neq 0$

A function is continuous if $L^- = L^+ = f(x)$ for all values of x.

First principles derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Not differentiable at:

- discontinuous points
- sharp point/cusp
- vertical tangents (∞ gradient)
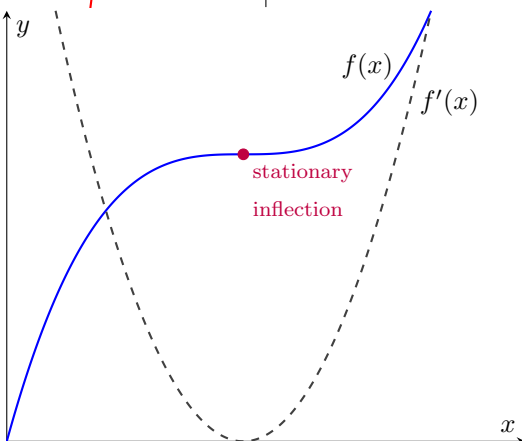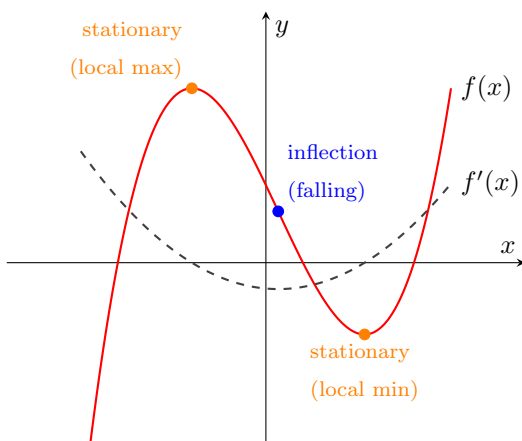
Solving on CAS

In main: type function. Interactive → Calculation → Line → (Normal | Tan line)

In graph: define function. Analysis → Sketch → (Normal | Tan line). Type x value to solve for a point. Return to show equation for line.

7   Statistics

Probability

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

$$\Pr(A \cap B) = \Pr(A|B) \times \Pr(B)$$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$\Pr(A) = \Pr(A|B) \cdot \Pr(B) + \Pr(A|B') \cdot \Pr(B')$$

Mutually exclusive $\implies \Pr(A \cup B) = 0$

Independent events:

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

$$\Pr(A|B) = \Pr(A)$$

$$\Pr(B|A) = \Pr(B)$$

Combinatorics

- Arrangements $\binom{n}{k} = \frac{n!}{(n-k)}$

- Combinations $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

- Note $\binom{n}{k} = \binom{n}{k-1}$

Distributions

Mean μ

Mean μ or **expected value** $E(X)$

$$E(X) = \frac{\Sigma\,[x \cdot f(x)]}{\Sigma f} \qquad (f = \text{absolute frequency})$$

$$= \sum_{i=1}^{n} [x_i \cdot \Pr(X = x_i)] \qquad (\text{discrete})$$

$$= \int_{\mathbf{X}} (x \cdot f(x))\,dx$$

Mode

Most popular value (has highest probability of all X values). Multiple modes can exist if > 1 X value have equal-highest probability. Number must exist in distribution.

Median

If $m > 0.5$, then value of X that is reached is the median of X. If $m = 0.5 = 0.5$, then m is halfway

Stationary points

Stationary point: $f'(x) = 0$

Point of inflection: $f'' = 0$

between this value and the next. To find m, add values of X from smallest to alrgest until the sum reaches 0.5.

$$m = X \text{ such that } \int_{-\infty}^{m} f(x)dx = 0.5$$

Variance σ^2

$$\text{Var}(x) = \sum_{i=1}^{n} p_i(x_i - \mu)^2$$
$$= \sum(x - \mu)^2 \times \Pr(X = x)$$
$$= \sum x^2 \times p(x) - \mu^2$$
$$= \text{E}(X^2) - [\text{E}(X)]^2 \qquad = E\left[(X - \mu)^2\right]$$

Standard deviation σ

$$\sigma = \text{sd}(X)$$
$$= \sqrt{\text{Var}(X)}$$

Binomial distributions

Conditions for a *binomial distribution*:

1. Two possible outcomes: **success** or **failure**

2. Pr(success) is constant across trials (also denoted p)

3. Finite number n of independent trials

Properties of $X \sim \text{Bi}(n, p)$

$$\mu(X) = np$$
$$\text{Var}(X) = np(1 - p)$$
$$\sigma(X) = \sqrt{np(1 - p)}$$
$$\Pr(X = x) = \binom{n}{x} \cdot p^x \cdot (1 - p)^{n-x}$$

> **On CAS**
>
> Interactive → Distribution → `binomialPdf` then input
>
> | | |
> |---|---|
> | x: | no. of successes |
> | numtrial: | no. of trials |
> | pos: | probability of success |

Continuous random variables

A continuous random variable X has a pdf f such that:

1. $f(x) \geq 0 \forall x$

2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$E(X) = \int_{\mathbf{X}} (x \cdot f(x))\, dx$$
$$\text{Var}(X) = E\left[(X - \mu)^2\right]$$

$$\Pr(X \leq c) = \int_{-\infty}^{c} f(x)\, dx$$

Two random variables X, Y

If X and Y are independent:
$$\text{E}(aX + bY) = a\,\text{E}(X) + b\,\text{E}(Y)$$
$$\text{Var}(aX \pm bY \pm c) = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y)$$

Linear functions $X \to aX + b$

$$\Pr(Y \leq y) = \Pr(aX + b \leq y)$$
$$= \Pr\left(X \leq \frac{y - b}{a}\right)$$
$$= \int_{-\infty}^{\frac{y-b}{a}} f(x)\, dx$$

Mean:	$\text{E}(aX + b) = a\,\text{E}(X) + b$
Variance:	$\text{Var}(aX + b) = a^2\,\text{Var}(X)$

Expectation theorems

For some non-linear function g, the expected value $E(g(X))$ is not equal to $g(E(X))$.

$$E(X^2) = \text{Var}(X) - [E(X)]^2$$
$$E(X^n) = \Sigma x^n \cdot p(x) \qquad \text{(non-linear)}$$
$$\neq [E(X)]^n$$
$$E(aX \pm b) = aE(X) \pm b \qquad \text{(linear)}$$
$$E(b) = b \qquad (\forall b \in \mathbb{R})$$
$$E(X + Y) = E(X) + E(Y) \qquad \text{(two variables)}$$

Sample mean

Approximation of the **population mean** determined experimentally.

$$\overline{x} = \frac{\Sigma x}{n}$$

where

  n is the size of the sample (number of sample points)

  x is the value of a sample point

> **On CAS**
>
> 1. Spreadsheet
>
> 2. In cell A1:
>
>   `mean(randNorm(sd, mean, sample size))`
>
> 3. Edit → Fill → Fill Range
>
> 4. Input range as A1:An where n is the number of samples
>
> 5. Graph → Histogram

Sample size of n

$$\overline{X} = \sum_{i=1}^{n} \frac{x_i}{n} = \frac{\sum x}{n}$$

Sample mean is distributed with mean μ and sd $\frac{\sigma}{\sqrt{n}}$ (approaches these values for increasing sample size n). For a new distribution with mean of n trials, $\mathrm{E}(X') = \mathrm{E}(X)$, $\mathrm{sd}(X') = \frac{\mathrm{sd}(X)}{\sqrt{n}}$

> **On CAS**
>
> - Spreadsheet   →   Catalog   → `randNorm(sd, mean, n)` where `n` is the number of samples. Show histogram with Histogram key in top left
>
> - To calculate parameters of a dataset: Calc → One-variable

Normal distributions

$$Z = \frac{X - \mu}{\sigma}$$

Normal distributions must have area (total prob.) of 1

$\implies \int_{-\infty}^{\infty} f(x)\, dx = 1$

mean = mode = median

> Always express z as +ve. Express confidence interval as ordered pair.