Polynomials

Quadratics

General form	$x^2 + bx + c = (x + m)(x + n)$ where $mn = c,\ m + n = b$
Difference of squares	$a^2 - b^2 = (a - b)(a + b)$
Perfect squares	$a^2 \pm 2ab + b^2 = (a \pm b^2)$
Completing the square	$x^2 + bx + c = (x + \frac{b}{2})^2 + c - \frac{b^2}{4}$ $ax^2 + bx + c = a(x - \frac{b}{2a})^2 + c - \frac{b^2}{4a}$
Quadratic formula	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $\Delta = b^2 - 4ac$

Cubics

Difference of cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
Sum of cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Perfect cubes: $a^3 \pm 3a^2b + 3ab^2 \pm b^3 = (a \pm b)^3$

$$y = a(bx - h)^3 + c$$

- $m = 0$ at *stationary point of inflection* (i.e. $(\frac{h}{b}, k)$)
- in form $y = (x - a)^2(x - b)$, local max at $x = a$, local min at $x = b$
- in form $y = a(x - b)(x - c)(x - d)$: x-intercepts at b, c, d
- in form $y = a(x - b)^2(x - c)$, touches x-axis at b, intercept at c

Linear and quadratic graphs

Forms of linear equations

$y = mx + c$ where m is gradient and c is y-intercept
$\frac{x}{a} + \frac{y}{b} = 1$ where m is gradient and (x_1, y_1) lies on the graph
$y - y_1 = m(x - x_1)$ where $(a, 0)$ and $(0, b)$ are x- and y-intercepts

Line properties

Parallel lines: $m_1 = m_2$
Perpendicular lines: $m_1 \times m_2 = -1$
Distance: $|\vec{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Quartic graphs

Forms of quadratic equations

$y = ax^4$
$y = a(x - b)(x - c)(x - d)(x - e)$
$y = ax^4 + cd^2(c \geq 0)$
$y = ax^2(x - b)(x - c)$
$y = a(x - b)^2(x - c)^2$
$y = a(x - b)(x - c)^3$

Simultaneous equations (linear)

- **Unique solution -** lines intersect at point
- **Infinitely many solutions -** lines are equal
- **No solution -** lines are parallel

Solving $\begin{cases} px + qy = a \\ rx + sy = b \end{cases}$ **for** $\{0, 1, \infty\}$ **solutions**

where all coefficients are known except for one, and a, b are known

1. Write as matrices: $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$
2. Find determinant of first matrix: $\Delta = ps - qr$
3. Let $\Delta = 0$ for number of solutions $\neq 1$ or let $\Delta \neq 0$ for one unique solution.
4. Solve determinant equation to find variable
   - *— for infinite/no solutions: —*
5. Substitute variable into both original equations
6. Rearrange equations so that LHS of each is the same
7. $\text{RHS}(1) = \text{RHS}(2) \implies (1) = (2)\ \forall x$ (∞ solns)
   $\text{RHS}(1) \neq \text{RHS}(2) \implies (1) \neq (2)\ \forall x$ (0 solns)

On CAS: Matrix → `det`

Solving $\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$

- Use elimination
- Generate two new equations with only two variables
- Rearrange & solve
- Substitute one variable into another equation to find another variable
- etc.

Inverse functions

Functions

- vertical line test
- each x value produces only one y value

One to one functions

- $f(x)$ is *one to one* if $f(a) \neq f(b)$ if $a, b \in \text{dom}(f)$ and $a \neq b$
  $\implies$ unique y for each x ($\sin x$ is not 1:1, x^3 is)
- horizontal line test
- if not one to one, it is many to one

Deriving f^{-1}

- if $f(g(x)) = x$, then g is the inverse of f
- reflection across $y - x$
- ran $f =$ dom f^{-1}, dom $f =$ ran f^{-1}
- inverse $\neq$ inverse *function* (i.e. inverse must pass vertical line test)
  $\implies f^{-1}(x)$ exists $\iff f(x)$ is one to one
- $f^{-1}(x) = f(x)$ intersections may lie on line $y = x$

Requirements for showing working for f^{-1}

1. start with *"let $y = f(x)$"*
2. must state *"take inverse"* for line where y and x are swapped
3. do all working in terms of $y = ...$
4. for square root, state $\pm$ solutions then show restricted
5. for inverse *function*, state in function notation

Transformations

Order of operations: DRT

dilations — reflections — translations

Transforming x^n to $a(x-h)^n+K$

- dilation factor of $|a|$ units parallel to y-axis or from x-axis
- if $a<0$, graph is reflected over x-axis
- translation of k units parallel to y-axis or from x-axis
- translation of h units parallel to x-axis or from y-axis
- for $(ax)^n$, dilation factor is $\frac{1}{a}$ parallel to x-axis or from y-axis
- when $0<|a|<1$, graph becomes closer to axis

Transforming $f(x)$ to $y=Af[n(x+c)]+b$

Applies to exponential, log, trig, e^x, polynomials.
Functions must be written in form $y=Af[n(x+c)]+b$

- dilation by factor $|A|$ from x-axis (if $A<0$, reflection across y-axis)
- dilation by factor $\frac{1}{n}$ from y-axis (if $n<0$, reflection across x-axis)
- translation of c units from y-axis (x-shift)
- translation of b units from x-axis (y-shift)

Dilations

Two pairs of equivalent processes for $y=f(x)$:

1. - Dilating from x-axis: $(x,y)\to(x,by)$
   - Replacing y with $\frac{y}{b}$ to obtain $y=bf(x)$
2. - Dilating from y-axis: $(x,y)\to(ax,y)$
   - Replacing x with $\frac{x}{a}$ to obtain $y=f(\frac{x}{a})$

For graph of $y=\frac{1}{x}$, horizontal & vertical dilations are equivalent (symmetrical). If $y=\frac{a}{x}$, graph is contracted rather than dilated.

Matrix transformations

Find new point (x',y'). Substitute these into original equation to find image with original variables (x,y).

Reflections

- Reflection **in** axis = reflection **over** axis = reflection **across** axis
- Translations do not change

Translations

For $y=f(x)$, these processes are equivalent:

- applying the translation $(x,y)\to(x+h,y+k)$ to the graph of $y=f(x)$
- replacing x with $x-h$ and y with $y-k$ to obtain $y-k=f(x-h)$

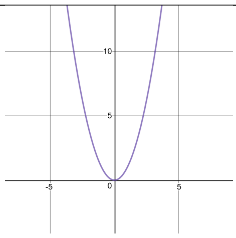
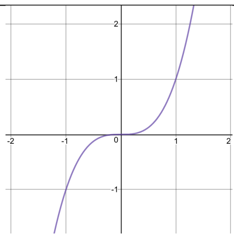
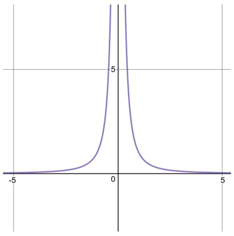
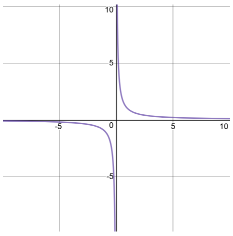
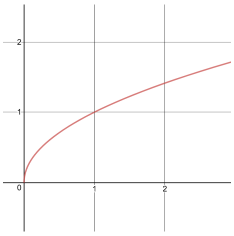
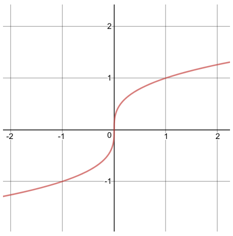
Power functions

Strictly increasing: $f(x_2)>f(x_1)$ where $x_2>x_1$ (including $x=0$)

Odd and even functions

Even when $f(x)=-f(x)$
Odd when $-f(x)=f(-x)$

Function is even if it can be reflected across y-axis $\implies f(x)=f(-x)$
Function $x^{\pm\frac{p}{q}}$ is odd if q is odd

	n is even	n is odd
x^n, $n \in \mathbb{Z}^+$		
x^n, $n \in \mathbb{Z}^-$		
$x^{\frac{1}{n}}$, $n \in \mathbb{Z}^+$		

$x^{\frac{-1}{n}}$ where $n \in \mathbb{Z}^+$

Mostly only on CAS.

We can write $x^{\frac{-1}{n}} = \frac{1}{x^{\frac{1}{n}}} = \frac{1}{\sqrt[n]{x}}$n.

Domain is: $\begin{cases} \mathbb{R} \setminus \{0\} & \text{if } n \text{ is odd} \\ \mathbb{R}^+ & \text{if } n \text{ is even} \end{cases}$

If n is odd, it is an odd function.

$x^{\frac{p}{q}}$ where $p, q \in \mathbb{Z}^+$

$$x^{\frac{p}{q}} = \sqrt[q]{x^p}$$

- if $p > q$, the shape of x^p is dominant
- if $p < q$, the shape of $x^{\frac{1}{q}}$ is dominant
- points $(0,0)$ and $(1,1)$ will always lie on graph
- Domain is: $\begin{cases} \mathbb{R} & \text{if } q \text{ is odd} \\ \mathbb{R}^+ \cup \{0\} & \text{if } q \text{ is even} \end{cases}$

Piecewise functions

$$\text{e.g.} \quad f(x) = \begin{cases} x^{1/3}, & x \le 0 \\ 2, & 0 < x < 2 \\ x, & x \ge 2 \end{cases}$$

Open circle: point included

Closed circle: point not included

Operations on functions

For $f \pm g$ and $f \times g$: $\quad \text{dom}' = \text{dom}(f) \cap \text{dom}(g)$

Addition of linear piecewise graphs: add y-values at key points

Product functions:

- product will equal 0 if $f = 0$ or $g = 0$
- $f'(x) = 0 \veebar g'(x) = 0 \not\Rightarrow (f \times g)'(x) = 0$

Composite functions

$(f \circ g)(x)$ is defined iff $\text{ran}(g) \subseteq \text{dom}(f)$

Exponentials & Logarithms

Index laws

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$(ab)^m = a^m b^m$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

$$\sqrt[n]{x} = x^{1/n}$$

Logarithm laws

$$\log_a(mn) = \log_a m + \log_a n$$

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a$$

$$\log_a(m^p) = p \log_a m$$

$$\log_a(m^{-1}) = -\log_a m$$

$$\log_a 1 = 0 \text{ and } \log_a a = 1$$

$$\log_b c = \frac{\log_a c}{\log_a b}$$

Inverse functions

For $f : \mathbb{R} \to \mathbb{R}, f(x) = a^x$, inverse is:

$$f^{-1} : \mathbb{R}^+ \to \mathbb{R}, f^{-1} = \log_a x$$

Exponentials

e^x natural exponential function

$$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

Modelling

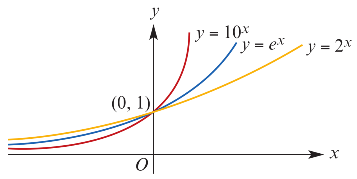
$$A = A_0 e^{kt}$$

- A_0 is initial value
- t is time taken
- k is a constant
- For continuous growth, $k > 0$
- For continuous decay, $k < 0$

Graphing exponential functions

$$f(x) = Aa^{k(x-b)} + c, \quad | \, a > 1$$

- **y-intercept** at $(0, A \cdot a^{-kb} + c)$ as $x \to \infty$
- **horizontal asymptote** at $y = c$
- **domain** is $\mathbb{R}$
- **range** is (c, ∞)
- dilation of factor $|A|$ from x-axis
- dilation of factor $\frac{1}{k}$ from y-axis



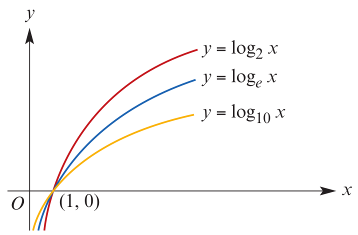
Graphing logarithmic functions

$\log_e x$ is the inverse of e^x (reflection across $y = x$)

$$f(x) = A \log_a k(x - b) + c$$

where

- **domain** is (b, ∞)
- **range** is $\mathbb{R}$
- **vertical asymptote** at $x = b$
- y-intercept exists if $b < 0$
- dilation of factor $|A|$ from x-axis
- dilation of factor $\frac{1}{k}$ from y-axis

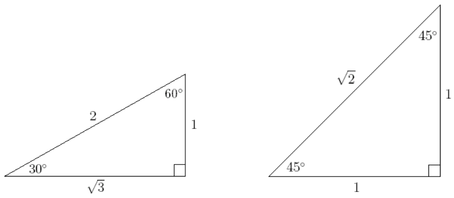


Finding equations

On CAS: $\begin{cases} f(3)=9 \\ g(3)=0 \end{cases} \Big|_{a,b}$

Circular functions

Exact values



$$1 \text{ rad} = \frac{180 \deg}{\pi}$$

$\sin$ and $\cos$ graphs

$$f(x) = a\sin(bx - c) + d$$

$$f(x) = a\cos(bx - c) + d$$

where

- a is the y-dilation (amplitude)
- b is the x-dilation (period)
- c is the x-shift (phase)
- d is the y-shift (equilibrium position)

Domain is $\mathbb{R}$

Range is $[-b + c, b + c]$;

Graph of $\cos(x)$ starts at $(0, 1)$. Graph of $\sin(x)$ starts at $(0, 0)$.

Mean / equilibrium: line that the graph oscillates around ($y = d$)

Amplitude

Graph oscillates between $+a$ and $-a$ in y-axis

$a = 0$ produces straight line

$a < 0$ inverts the phase (sin becomes cos, vice vera)

Period

Period T is $\frac{2\pi}{b}$

$b = 0$ produces straight line

$b < 0$ inverts the phase

Phase

c moves the graph left-right in the x axis.

If $c = T = \frac{2\pi}{b}$, the graph has no actual phase shift.

Symmetry

$$\sin(\theta + \frac{\pi}{2}) = \sin\theta$$

$$\sin(\theta + \pi) = -\sin\theta$$

$$\cos(\theta + \frac{\pi}{2}) = -\cos\theta$$

$$\cos(\theta + \pi) = -cos(\theta + \frac{3\pi}{2}) = \cos(-\theta)$$

Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

Complementary relationships

$$\sin(\frac{\pi}{2} - \theta) = \cos\theta$$

$$\cos(\frac{\pi}{2} - \theta) = \sin\theta$$

$$\sin\theta = -\cos(\theta + \frac{\pi}{2})$$

$$\cos\theta = \sin(\theta + \frac{\pi}{2})$$

$\tan$ graph

$$y = a\tan(nx)$$

where

- a is x-dilation (period)
- n is y-dilation ($\equiv$ amplitude)
- period T is $\frac{\pi}{n}$
- range is R
- roots at $x = \frac{k\pi}{n}$
- asymptotes at $x = \frac{(2k+1)\pi}{2n}, \quad k \in \mathbb{Z}$

Asymptotes should always have equations and arrow pointing up

Solving trig equations

1. Solve domain for $n\theta$
2. Find solutions for $n\theta$
3. Divide solutions by n

$\sin 2\theta = \frac{\sqrt{3}}{2}, \quad \theta \in [0, 2\pi] \quad (\therefore 2\theta \in [0, 4\pi])$

$2\theta = \sin^{-1}\frac{\sqrt{3}}{2}$

$2\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$

$\therefore \theta = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$

Calculus

Average rate of change

$$m \text{ of } x \in [a, b] = \frac{f(b) - f(a)}{b - a} = \frac{dy}{dx}$$

On CAS: Action → Calculation → `diff`

Instantaneous rate of change

Secant - line passing through two points on a curve
Chord - line segment joining two points on a curve

Limit theorems

1. For constant function $f(x) = k$, $\lim_{x \to a} f(x) = k$
2. $\lim_{x \to a}(f(x) \pm g(x)) = F \pm G$
3. $\lim_{x \to a}(f(x) \times g(x)) = F \times G$
4. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G}, G \neq 0$

A function is continuous if $L^- = L^+ = f(x)$ for all values of x.

First principles derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Not differentiable at:

- discontinuous points
- sharp point/cusp
- vertical tangents (∞ gradient)

Tangents & gradients

Tangent line - defined by $y = mx + c$ where $m = \frac{dy}{dx}$
Normal line - $\perp$ tangent ($m_{tan} \cdot m_{\text{norm}} = -1$)
Secant $= \frac{f(x+h) - f(x)}{h}$

Strictly increasing/decreasing

For x_2 and x_1 where $x_2 > x_1$:

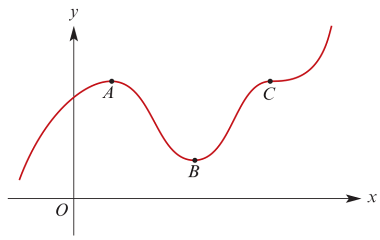
- **strictly increasing** where $f(x_2) > f(x_1)$ or $f'(x) > 0$
- **strictly decreasing** where $f(x_2) < f(x_1)$ or $f'(x) < 0$
- Endpoints are included, even where gradient $= 0$

Solving on CAS

In main : type function. Interactive → Calculation → Line → (Normal | Tan line)
In graph : define function. Analysis → Sketch → (Normal | Tan line). Type x value to solve for a point. Return to show equation for line.

Stationary points

Stationary where $m = 0$.
Find derivative, solve for $\frac{dy}{dx} = 0$



Local maximum at point A

- $f'(x) > 0$ left of A
- $f'(x) < 0$ right of A

Local minimum at point B

- $f'(x) < 0$ left of B
- $f'(x) > 0$ right of B

Stationary point of inflection at C

Function derivatives

$f(x)$	$f'(x)$
kx^n	knx^{n-1}
$g(x) \pm h(x)$	$g'(x) \pm h'(x)$
c	0
$\frac{u}{v}$	$(v\frac{du}{dx} - u\frac{dv}{dx}) \div v^2$
uv	$u\frac{dv}{dx} + v\frac{du}{dx}$
$f \circ g$	$\frac{dy}{du} \cdot \frac{du}{dx}$
$\sin ax$	$a \cos ax$
$\sin(f(x))$	$f'(x) \cdot \cos(f(x))$
$\cos ax$	$-a \sin ax$
$\cos(f(x))$	$f'(x)(-\sin(f(x)))$
e^{ax}	ae^{ax}
$\log_e ax$	$\frac{1}{x}$
$\log_e f(x)$	$\frac{f'(x)}{f(x)}$