

Year 12 Methods

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1 Functions

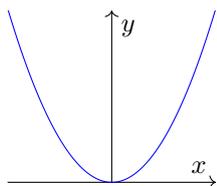
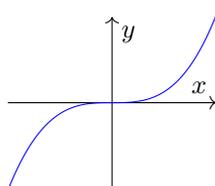
- vertical line test
- each x value produces only one y value

One to one functions

- $f(x)$ is *one to one* if $f(a) \neq f(b)$ if $a, b \in \text{dom}(f)$ and $a \neq b$
 \implies unique y for each x ($\sin x$ is not 1:1, x^3 is)
- horizontal line test
- if not one to one, it is many to one

Odd and even functions

$$\begin{array}{ll} \text{Even:} & f(x) = f(-x) \\ \text{Odd:} & -f(x) = f(-x) \end{array}$$

Even \implies symmetrical across y -axis $x^{\pm \frac{p}{q}}$ is odd if q is oddFor x^n , parity of $n \equiv$ parity of function**Even:****Odd:**

Inverse functions

- Inverse of $f(x)$ is denoted $f^{-1}(x)$
- f must be one to one
- If $f(g(x)) = x$, then g is the inverse of f
- Represents reflection across $y = x$
- $\implies f^{-1}(x) = f(x)$ intersections lie on $y = x$
- $\text{ran } f = \text{dom } f^{-1}$
 $\text{dom } f = \text{ran } f^{-1}$
- “Inverse” \neq “inverse function” (functions must pass vertical line test)

Finding f^{-1}

1. Let $y = f(x)$
2. Swap x and y (“take inverse”)
3. Solve for y
 Sqrt: state \pm solutions then restrict
4. State rule as $f^{-1}(x) = \dots$
5. For inverse *function*, state in function notation

Simultaneous equations (linear)

- **Unique solution** - lines intersect at point
- **Infinitely many solutions** - lines are equal
- **No solution** - lines are parallel

$$\text{Solving } \begin{cases} px + qy = a \\ rx + sy = b \end{cases} \quad \text{for } \{0, 1, \infty\} \text{ solutions}$$

where all coefficients are known except for one, and a, b are known

1. Write as matrices: $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$
2. Find determinant of first matrix: $\Delta = ps - qr$
3. Let $\Delta = 0$ for number of solutions $\neq 1$
 or let $\Delta \neq 0$ for one unique solution.
4. Solve determinant equation to find variable
For infinite/no solutions:
5. Substitute variable into both original equations
6. Rearrange equations so that LHS of each is the same
7. $\text{RHS}(1) = \text{RHS}(2) \implies (1) = (2) \forall x$ (∞ solns)
 $\text{RHS}(1) \neq \text{RHS}(2) \implies (1) \neq (2) \forall x$ (0 solns)

On CAS: Matrix \rightarrow det

$$\text{Solving } \begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

- Use elimination

- Generate two new equations with only two variables
- Rearrange & solve
- Substitute one variable into another equation to find another variable

Piecewise functions

$$\text{e.g. } f(x) = \begin{cases} x^{1/3}, & x \leq 0 \\ 2, & 0 < x < 2 \\ x, & x \geq 2 \end{cases}$$

Open circle: point included

Closed circle: point not included

Operations on functions

For $f \pm g$ and $f \times g$: $\text{dom}' = \text{dom}(f) \cap \text{dom}(g)$

Addition of linear piecewise graphs: add y -values at key points

Product functions:

- product will equal 0 if $f = 0$ or $g = 0$
- $f'(x) = 0 \vee g'(x) = 0 \not\Rightarrow (f \times g)'(x) = 0$

Composite functions

$(f \circ g)(x)$ is defined iff $\text{ran}(g) \subseteq \text{dom}(f)$

2 Polynomials

Linear equations

Forms

- $y = mx + c$
- $\frac{x}{a} + \frac{y}{b} = 1$ where (x_1, y_1) lies on the graph
- $y - y_1 = m(x - x_1)$ where $(a, 0)$ and $(0, b)$ are x - and y -intercepts

Line properties

Parallel lines: $m_1 = m_2$

Perpendicular lines: $m_1 \times m_2 = -1$

Distance: $|\vec{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Quadratics

$$x^2 + bx + c = (x + m)(x + n)$$

where $mn = c, m + n = b$

Difference of squares

$$a^2 - b^2 = (a - b)(a + b)$$

Perfect squares

$$a^2 \pm 2ab + b^2 = (a \pm b)^2$$

Completing the square

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}$$

$$ax^2 + bx + c = a\left(x - \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(Discriminant $\Delta = b^2 - 4ac$)

Cubics

Difference of cubes

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Sum of cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Perfect cubes

$$a^3 \pm 3a^2b + 3ab^2 \pm b^3 = (a \pm b)^3$$

$$y = a(bx - h)^3 + c$$

- $m = 0$ at *stationary point of inflection* (i.e. $(\frac{h}{b}, k)$)
- $y = (x - a)^2(x - b)$ — max at $x = a$, min at $x = b$
- $y = a(x - b)(x - c)(x - d)$ — roots at b, c, d
- $y = a(x - b)^2(x - c)$ — roots at b (instantaneous), c (intercept)

Quartic graphs

Forms of quartic equations

$$y = ax^4$$

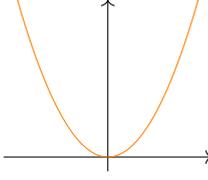
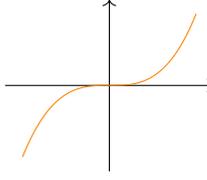
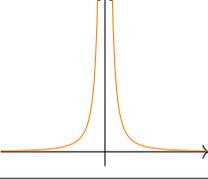
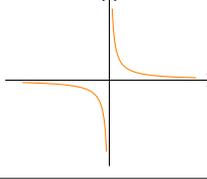
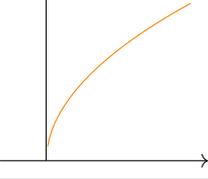
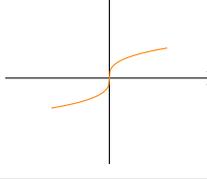
$$y = a(x - b)(x - c)(x - d)(x - e)$$

$$y = ax^4 + cd^2 (c \geq 0)$$

$$y = ax^2(x - b)(x - c)$$

$$y = a(x - b)^2(x - c)^2$$

$$y = a(x - b)(x - c)^3$$

	n is even	n is odd
$x^n, n \in \mathbb{Z}^+$		
$x^n, n \in \mathbb{Z}^-$		
$x^{\frac{1}{n}}, n \in \mathbb{Z}^-$		

3 Transformations

- translation of b units from x -axis (y -shift)

Order of operations: DRT

dilations — reflections — translations

Transforming x^n to $a(x - h)^n + K$

- dilation factor of $|a|$ units parallel to y -axis or from x -axis
- if $a < 0$, graph is reflected over x -axis
- translation of k units parallel to y -axis or from x -axis
- translation of h units parallel to x -axis or from y -axis
- for $(ax)^n$, dilation factor is $\frac{1}{a}$ parallel to x -axis or from y -axis
- when $0 < |a| < 1$, graph becomes closer to axis

Transforming $f(x)$ to $y = Af[n(x + c)] + b$

Applies to exponential, log, trig, e^x , polynomials.

Functions must be written in form $y = Af[n(x + c)] + b$

- dilation by factor $|A|$ from x -axis (if $A < 0$, reflection across y -axis)
- dilation by factor $\frac{1}{n}$ from y -axis (if $n < 0$, reflection across x -axis)
- translation of c units from y -axis (x -shift)

Dilations

Two pairs of equivalent processes for $y = f(x)$:

- Dilating from x -axis: $(x, y) \rightarrow (x, by)$
 - Replacing y with $\frac{y}{b}$ to obtain $y = bf(x)$
- Dilating from y -axis: $(x, y) \rightarrow (ax, y)$
 - Replacing x with $\frac{x}{a}$ to obtain $y = f(\frac{x}{a})$

For graph of $y = \frac{1}{x}$, horizontal & vertical dilations are equivalent (symmetrical). If $y = \frac{a}{x}$, graph is contracted rather than dilated.

Matrix transformations

Find new point (x', y') . Substitute these into original equation to find image with original variables (x, y) .

Reflections

- Reflection **in** axis = reflection **over** axis = reflection **across** axis
- Translations do not change

Translations

For $y = f(x)$, these processes are equivalent:

- applying the translation $(x, y) \rightarrow (x + h, y + k)$ to the graph of $y = f(x)$
- replacing x with $x - h$ and y with $y - k$ to obtain $y - k = f(x - h)$

Power functions

Mostly only on CAS.

We can write $x^{-\frac{1}{n}} = \frac{1}{x^{\frac{1}{n}}} = \frac{1}{\sqrt[n]{x}}$.

Domain is: $\begin{cases} \mathbb{R} \setminus \{0\} & \text{if } n \text{ is odd} \\ \mathbb{R}^+ & \text{if } n \text{ is even} \end{cases}$

If n is odd, it is an odd function.

$x^{\frac{p}{q}}$ where $p, q \in \mathbb{Z}^+$

$$x^{\frac{p}{q}} = \sqrt[q]{x^p}$$

- if $p > q$, the shape of x^p is dominant
- if $p < q$, the shape of $x^{\frac{1}{q}}$ is dominant
- points $(0, 0)$ and $(1, 1)$ will always lie on graph
- Domain is: $\begin{cases} \mathbb{R} & \text{if } q \text{ is odd} \\ \mathbb{R}^+ \cup \{0\} & \text{if } q \text{ is even} \end{cases}$

4 Exponentials & Logarithms

Logarithmic identities

$$\begin{aligned} \log_b(xy) &= \log_b x + \log_b y \\ \log_b x^n &= n \log_b x \\ \log_b y^{x^n} &= x^n \log_b y \\ \log_a\left(\frac{m}{n}\right) &= \log_a m - \log_a n \\ \log_a(m^{-1}) &= -\log_a m \\ \log_b c &= \frac{\log_a c}{\log_a b} \end{aligned}$$

Index identities

$$\begin{aligned} b^{m+n} &= b^m \cdot b^n \\ (b^m)^n &= b^{m \cdot n} \\ (b \cdot c)^n &= b^n \cdot c^n \\ b^m \div a^n &= b^{m-n} \end{aligned}$$

Inverse functions

For $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = a^x$, inverse is:

$$f^{-1} : \mathbb{R}^+ \rightarrow \mathbb{R}, f^{-1} = \log_a x$$

Euler's number e

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

Modelling

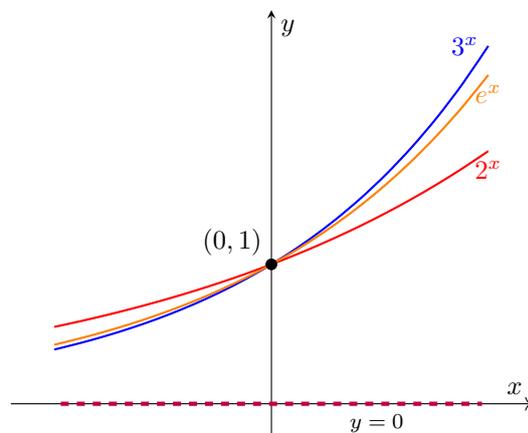
$$A = A_0 e^{kt}$$

- A_0 is initial value
- t is time taken
- k is a constant
- For continuous growth, $k > 0$
- For continuous decay, $k < 0$

Graphing exponential functions

$$f(x) = Aa^{k(x-b)} + c, \quad |a > 1$$

- **y-intercept** at $(0, A \cdot a^{-kb} + c)$ as $x \rightarrow \infty$
- **horizontal asymptote** at $y = c$
- **domain** is \mathbb{R}
- **range** is (c, ∞)
- dilation of factor $|A|$ from x -axis
- dilation of factor $\frac{1}{k}$ from y -axis



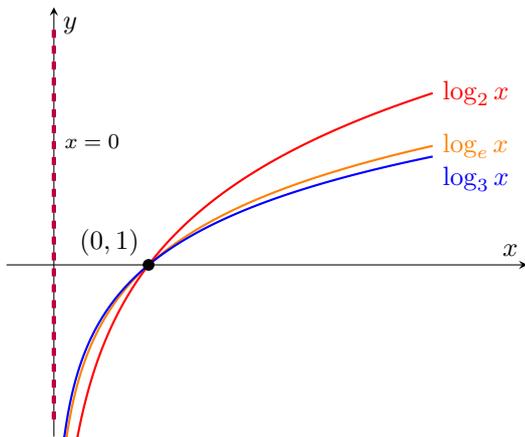
Graphing logarithmic functions

$\log_e x$ is the inverse of e^x (reflection across $y = x$)

$$f(x) = A \log_a k(x - b) + c$$

where

- **domain** is (b, ∞)
- **range** is \mathbb{R}
- **vertical asymptote** at $x = b$
- y -intercept exists if $b < 0$
- dilation of factor $|A|$ from x -axis
- dilation of factor $\frac{1}{k}$ from y -axis



Finding equations

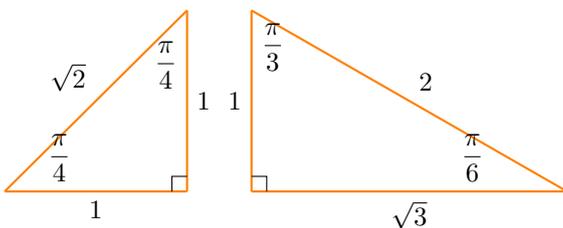
On CAS: $\begin{cases} f(3)=9 \\ g(3)=0 \end{cases} \Big|_{a,b}$

5 Circular functions

Radians and degrees

$$1 \text{ rad} = \frac{180 \text{ deg}}{\pi}$$

Exact values



Compound angle formulas

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

Double angle formulas

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= 1 - 2 \sin^2 x$$

$$= 2 \cos^2 x - 1$$

$$\sin 2x = 2 \sin x \cos x$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Symmetry

$$\sin(\theta + \frac{\pi}{2}) = \cos \theta$$

$$\sin(\theta + \pi) = -\sin \theta$$

$$\cos(\theta + \frac{\pi}{2}) = -\sin \theta$$

$$\begin{aligned} \cos(\theta + \pi) &= -\cos(\theta + \frac{3\pi}{2}) \\ &= \cos(-\theta) \end{aligned}$$

Complementary relationships

$$\sin \theta = \cos(\frac{\pi}{2} - \theta)$$

$$= -\cos(\theta + \frac{\pi}{2})$$

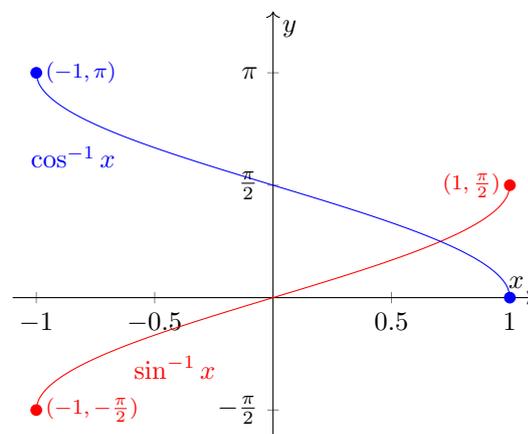
$$\cos \theta = \sin(\frac{\pi}{2} - \theta)$$

$$= \sin(\theta + \frac{\pi}{2})$$

Pythagorean identity

$$\cos^2 \theta + \sin^2 \theta = 1$$

Inverse circular functions



Inverse functions: $f(f^{-1}(x)) = x$ (restrict domain)

$$\sin^{-1} : [-1, 1] \rightarrow \mathbb{R}, \quad \sin^{-1} x = y$$

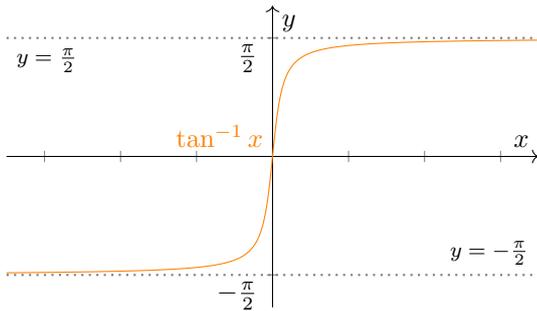
$$\text{where } \sin y = x, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\cos^{-1} : [-1, 1] \rightarrow \mathbb{R}, \quad \cos^{-1} x = y$$

$$\text{where } \cos y = x, y \in [0, \pi]$$

$$\tan^{-1} : \mathbb{R} \rightarrow \mathbb{R}, \quad \tan^{-1} x = y$$

$$\text{where } \tan y = x, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$



sin and cos graphs

$$f(x) = a \sin(bx - c) + d$$

where:

$$\text{Period} = \frac{2\pi}{n}$$

$$\text{dom} = \mathbb{R}$$

$$\text{ran} = [-b + c, b + c];$$

$\cos(x)$ starts at $(0, 1)$, $\sin(x)$ starts at $(0, 0)$

0 amplitude \implies straight line

$a < 0$ or $b < 0$ inverts phase (swap sin and cos)

$$c = T = \frac{2\pi}{b} \implies \text{no net phase shift}$$

tan graphs

$$y = a \tan(nx)$$

$$\text{Period} = \frac{\pi}{n}$$

Range is \mathbb{R}

$$\text{Roots at } x = \frac{k\pi}{n} \text{ where } k \in \mathbb{Z}$$

$$\text{Asymptotes at } x = \frac{(2k+1)\pi}{2n}$$

Asymptotes should always have equations

Solving trig equations

1. Solve domain for $n\theta$
2. Find solutions for $n\theta$
3. Divide solutions by n

$$\sin 2\theta = \frac{\sqrt{3}}{2}, \quad \theta \in [0, 2\pi] \quad (\because 2\theta \in [0, 4\pi])$$

$$2\theta = \sin^{-1} \frac{\sqrt{3}}{2}$$

$$2\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$$

$$\therefore \theta = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$$

6 Calculus

Average rate of change

$$m \text{ of } x \in [a, b] = \frac{f(b) - f(a)}{b - a} = \frac{dy}{dx}$$

On CAS: Action \rightarrow Calculation \rightarrow diff

Average value

$$f_{\text{avg}} = \frac{1}{b - a} \int_a^b f(x) dx$$

Instantaneous rate of change

Secant - line passing through two points on a curve

Chord - line segment joining two points on a curve

Limit theorems

1. For constant function $f(x) = k, \lim_{x \rightarrow a} f(x) = k$
2. $\lim_{x \rightarrow a} (f(x) \pm g(x)) = F \pm G$
3. $\lim_{x \rightarrow a} (f(x) \times g(x)) = F \times G$
4. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{F}{G}, G \neq 0$

A function is continuous if $L^- = L^+ = f(x)$ for all values of x .

First principles derivative

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Not differentiable at:

- discontinuous points
- sharp point/cusp
- vertical tangents (∞ gradient)

Tangents & gradients

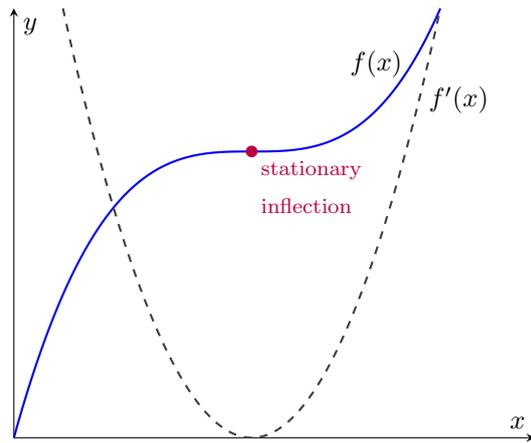
Tangent line - defined by $y = mx + c$ where $m = \frac{dy}{dx}$

Normal line - \perp tangent ($m_{tan} \cdot m_{norm} = -1$)

Secant = $\frac{f(x+h)-f(x)}{h}$

On CAS:

Action \rightarrow Calculation \rightarrow Line \rightarrow tanLine or normal



Strictly increasing/decreasing

For x_2 and x_1 where $x_2 > x_1$:

- **strictly increasing**
where $f(x_2) > f(x_1)$ or $f'(x) > 0$
- **strictly decreasing**
where $f(x_2) < f(x_1)$ or $f'(x) < 0$
- Endpoints are included, even where gradient = 0

On CAS

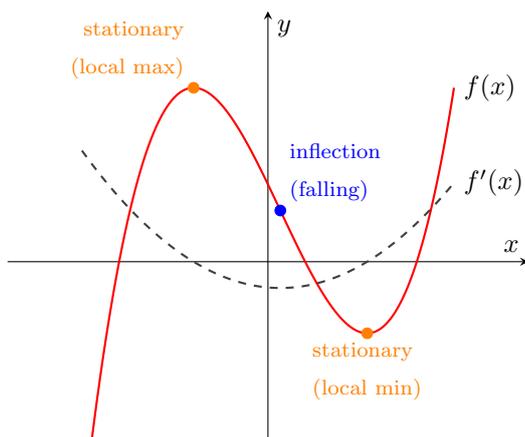
In main: type function. Interactive \rightarrow Calculation \rightarrow Line \rightarrow (Normal | Tan line)

In graph: define function. Analysis \rightarrow Sketch \rightarrow (Normal | Tan line). Type x value to solve for a point. Return to show equation for line.

Stationary points

Stationary point: $f'(x) = 0$

Point of inflection: $f'' = 0$



Derivatives

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\sin ax$	$a \cos ax$
$\cos x$	$-\sin x$
$\cos ax$	$-a \sin ax$
$\tan f(x)$	$f^2(x) \sec^2 f(x)$
e^x	e^x
e^{ax}	ae^{ax}
ax^{nx}	$an \cdot e^{nx}$
$\log_e x$	$\frac{1}{x}$
$\log_e ax$	$\frac{1}{x}$
$\log_e f(x)$	$\frac{f'(x)}{f(x)}$
$\sin(f(x))$	$f'(x) \cdot \cos(f(x))$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\frac{d}{dy} f(y)$	$\frac{1}{\frac{dy}{dx}}$ (reciprocal)
uv	$u \frac{dv}{dx} + v \frac{du}{dx}$ (product rule)
$\frac{u}{v}$	$\frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$ (quotient rule)
$f(g(x))$	$f'(g(x)) \cdot g'(x)$

Antiderivatives

$f(x)$	$\int f(x) \cdot dx$
k (constant)	$kx + c$
x^n	$\frac{1}{n+1}x^{n+1}$
ax^{-n}	$a \cdot \log_e x + c$
$\frac{1}{ax+b}$	$\frac{1}{a} \log_e(ax+b) + c$
$(ax+b)^n$	$\frac{1}{a(n+1)}(ax+b)^{n+1} + c \mid n \neq -1$
$(ax+b)^{-1}$	$\frac{1}{a} \log_e ax+b + c$
e^{kx}	$\frac{1}{k}e^{kx} + c$
e^k	$e^kx + c$
$\sin kx$	$-\frac{1}{k} \cos(kx) + c$
$\cos kx$	$\frac{1}{k} \sin(kx) + c$
$\sec^2 kx$	$\frac{1}{k} \tan(kx) + c$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1} \frac{x}{a} + c \mid a > 0$
$\frac{-1}{\sqrt{a^2-x^2}}$	$\cos^{-1} \frac{x}{a} + c \mid a > 0$
$\frac{a}{a^2-x^2}$	$\tan^{-1} \frac{x}{a} + c$
$\frac{f'(x)}{f(x)}$	$\log_e f(x) + c$
$\int f(u) \cdot \frac{du}{dx} \cdot dx$	$\int f(u) \cdot du$ (substitution)
$f(x) \cdot g(x)$	$\int [f'(x) \cdot g(x)]dx + \int [g'(x)f(x)]dx$

7 Statistics

Probability

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

$$\Pr(A \cap B) = \Pr(A|B) \times \Pr(B)$$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$\Pr(A) = \Pr(A|B) \cdot \Pr(B) + \Pr(A|B') \cdot \Pr(B')$$

Mutually exclusive: $\Pr(A \cap B) = 0$

Independent events:

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

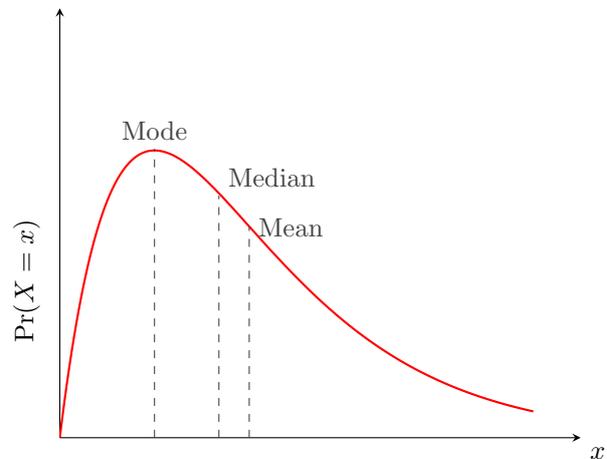
$$\Pr(A|B) = \Pr(A)$$

$$\Pr(B|A) = \Pr(B)$$

Combinatorics

- Arrangements $\binom{n}{k} = \frac{n!}{(n-k)!}$
- **Combinations** $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- Note $\binom{n}{k} = \binom{n}{n-k}$

Distributions



Mean μ

$$E(X) = \frac{\sum [x \cdot f(x)]}{\sum f} \quad (f = \text{absolute frequency})$$

$$= \sum_{i=1}^n [x_i \cdot \Pr(X = x_i)] \quad (\text{discrete})$$

$$= \int_{\mathbf{X}} (x \cdot f(x)) dx$$

Mode

Value of X which has the highest probability

- Most popular value in discrete distributions
- Must exist in distribution
- Represented by local max in pdf
- Multiple modes exist when > 1 X value have equal-highest probability

Median

Value separating lower and upper half of distribution

Continuous:

$$m = X \text{ such that } \int_{-\infty}^m f(x) dx = 0.5$$

Discrete: (not in course)

- Does not have to exist in distribution
- Add values of X smallest to largest until sum is ≥ 0.5
- If $X_1 < 0.5 < X_2$, then median is the average of X_1 and X_2
 - If $m > 0.5$, then value of X that is reached is the median of X

Variance σ^2

$$\begin{aligned} \text{Var}(x) &= \sum_{i=1}^n p_i (x_i - \mu)^2 \\ &= \sum (x - \mu)^2 \times \text{Pr}(X = x) \\ &= \sum x^2 \times p(x) - \mu^2 \\ &= E(X^2) - [E(X)]^2 \\ &= E[(X - \mu)^2] \end{aligned}$$

Standard deviation σ

$$\begin{aligned} \sigma &= \text{sd}(X) \\ &= \sqrt{\text{Var}(X)} \end{aligned}$$

Binomial distributions

Conditions for a *binomial distribution*:

1. Two possible outcomes: **success** or **failure**
2. $\text{Pr}(\text{success}) (=p)$ is constant across trials
3. Finite number n of independent trials

Properties of $X \sim \text{Bi}(n, p)$

$$\begin{aligned} \mu(X) &= np \\ \text{Var}(X) &= np(1 - p) \\ \sigma(X) &= \sqrt{np(1 - p)} \\ \text{Pr}(X = x) &= \binom{n}{x} \cdot p^x \cdot (1 - p)^{n-x} \end{aligned}$$

On CAS

Interactive \rightarrow Distribution \rightarrow binomialPdf

x:	no. of successes
numtrial:	no. of trials
pos:	probability of success

Continuous random variables

A continuous random variable X has a pdf f such that:

1. $f(x) \geq 0 \forall x$
2. $\int_{-\infty}^{\infty} f(x) dx = 1$

$$E(X) = \int_{\mathbf{x}} (x \cdot f(x)) dx$$

$$\text{Var}(X) = E[(X - \mu)^2]$$

$$\text{Pr}(X \leq c) = \int_{-\infty}^c f(x) dx$$

On CAS

Define piecewise functions:

Math3 \rightarrow

Two random variables X, Y

If X and Y are independent:

$$E(aX + bY) = aE(X) + bE(Y)$$

$$\text{Var}(aX \pm bY \pm c) = a^2 \text{Var}(X) + b^2 \text{Var}(Y)$$

Linear functions $X \rightarrow aX + b$

$$\text{Pr}(Y \leq y) = \text{Pr}(aX + b \leq y)$$

$$= \text{Pr}\left(X \leq \frac{y - b}{a}\right)$$

$$= \int_{-\infty}^{\frac{y-b}{a}} f(x) dx$$

Mean: $E(aX + b) = aE(X) + b$

Variance: $\text{Var}(aX + b) = a^2 \text{Var}(X)$

Expectation theorems

For some non-linear function g , the expected value $E(g(X))$ is not equal to $g(E(X))$.

$$E(X^2) = \text{Var}(X) + [E(X)]^2$$

$$E(X^n) = \sum x^n \cdot p(x) \quad (\text{non-linear})$$

$$\neq [E(X)]^n$$

$$E(aX \pm b) = aE(X) \pm b \quad (\text{linear})$$

$$E(b) = b \quad (\forall b \in \mathbb{R})$$

$$E(X + Y) = E(X) + E(Y) \quad (\text{two variables})$$

Sample mean

Approximation of the **population mean** determined experimentally.

$$\bar{x} = \frac{\sum x}{n}$$

where

n is the size of the sample (number of sample points)

x is the value of a sample point

On CAS

1. Spreadsheet
2. In cell A1:
`mean(randNorm(sd, mean, sample size))`
3. Edit → Fill → Fill Range
4. Input range as A1:An where n is the number of samples
5. Graph → Histogram

Sample size of n

$$\bar{X} = \sum_{i=1}^n \frac{x_i}{n} = \frac{\sum x}{n}$$

Sample mean is distributed with mean μ and sd $\frac{\sigma}{\sqrt{n}}$ (approaches these values for increasing sample size n).

For a new distribution with mean of n trials, $E(X') = E(X)$, $\text{sd}(X') = \frac{\text{sd}(X)}{\sqrt{n}}$

On CAS

- Spreadsheet → Catalog → `randNorm(sd, mean, n)` where n is the number of samples. Show histogram with Histogram key in top left
- To calculate parameters of a dataset: Calc → One-variable

Normal distributions

$$Z = \frac{X - \mu}{\sigma}$$

Normal distributions must have area (total prob.) of 1

$$\implies \int_{-\infty}^{\infty} f(x) dx = 1$$

mean = mode = median

Always express z as +ve. Express confidence interval as ordered pair.

Confidence intervals

- **Point estimate:** single-valued estimate of the population mean from the value of the sample mean \bar{x}
- **Interval estimate:** confidence interval for population mean μ
- $C\%$ confidence interval $\implies C\%$ of samples will contain population mean μ

95% confidence interval

For 95% c.i. of population mean μ :

$$x \in \left(\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}} \right)$$

where:

- \bar{x} is the sample mean
- σ is the population sd
- n is the sample size from which \bar{x} was calculated

On CAS
 Menu → Stats → Calc → Interval
 Set *Type* = *One-Sample Z Int*
 and select *Variable*

General case

For $C\%$ c.i. of population mean μ :

$$x \in \left(\bar{x} \pm k \frac{\sigma}{\sqrt{n}} \right)$$

where k is such that $\Pr(-k < Z < k) = \frac{C}{100}$

On CAS
 Menu → Stats → Calc → Interval
 Set *Type* = *One-Prop Z Int*
 Input $x = \hat{p} * n$

Margin of error

For 95% confidence interval of μ :

$$\begin{aligned} M &= 1.96 \times \frac{\sigma}{\sqrt{n}} \\ &= \frac{1}{2} \times \text{width of c.i.} \\ \Rightarrow n &= \left(\frac{1.96\sigma}{M} \right)^2 \end{aligned}$$

Always round n up to a whole number of samples.

Confidence interval for multiple trials

For a set of n confidence intervals (samples), there is 0.95^n chance that all n intervals contain the population mean μ .

