# 1 Statistics

## Continuous random variables

A continuous random variable $X$ has a pdf $f$ such that:

1. $f(x) \geq 0 \forall x$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$
\begin{aligned}
E(X) &= \int_{\mathbf{X}} (x \cdot f(x))\,dx \\
\operatorname{Var}(X) &= E\left[(X-\mu)^2\right]
\end{aligned}
$$

$$\Pr(X \leq c) = \int_{-\infty}^{c} f(x)\,dx$$

## Two random variables $X, Y$

If $X$ and $Y$ are independent:

$$
\begin{aligned}
\mathrm{E}(aX + bY) &= a\,\mathrm{E}(X) + b\,\mathrm{E}(Y) \\
\operatorname{Var}(aX \pm bY \pm c) &= a^2 \operatorname{Var}(X) + b^2 \operatorname{Var}(Y)
\end{aligned}
$$

## Linear functions $X \to aX + b$

$$
\begin{aligned}
\Pr(Y \leq y) &= \Pr(aX + b \leq y) \\
&= \Pr\left(X \leq \frac{y-b}{a}\right) \\
&= \int_{-\infty}^{\frac{y-b}{a}} f(x)\,dx
\end{aligned}
$$

**Mean:** $\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b$

**Variance:** $\operatorname{Var}(aX + b) = a^2 \operatorname{Var}(X)$

## Expectation theorems

For some non-linear function $g$, the expected value $E(g(X))$ is not equal to $g(E(X))$.

$$E(X^2) = \text{Var}(X) - [E(X)]^2$$

$$E(X^n) = \Sigma x^n \cdot p(x) \quad \text{(non-linear)}$$

$$\neq [E(X)]^n$$

$$E(aX \pm b) = aE(X) \pm b \quad \text{(linear)}$$

$$E(b) = b \quad (\forall b \in \mathbb{R})$$

$$E(X+Y) = E(X) + E(Y) \quad \text{(two variables)}$$

## Sample mean

Approximation of the **population mean** determined experimentally.

$$\overline{x} = \frac{\Sigma x}{n}$$

where

$n$ is the size of the sample (number of sample points)

$x$ is the value of a sample point

On CAS

1. Spreadsheet
2. In cell A1:

   `mean(randNorm(sd, mean, sample size))`
3. Edit → Fill → Fill Range
4. Input range as A1:An where $n$ is the number of samples
5. Graph → Histogram

### Sample size of $n$

$$\overline{X} = \sum_{i=1}^{n} \frac{x_i}{n} = \frac{\sum x}{n}$$

Sample mean is distributed with mean $\mu$ and sd $\frac{\sigma}{\sqrt{n}}$ (approaches these values for increasing sample size $n$).

For a new distribution with mean of $n$ trials, $\mathrm{E}(X') = \mathrm{E}(X), \quad \mathrm{sd}(X') = \dfrac{\mathrm{sd}(X)}{\sqrt{n}}$

On CAS

- Spreadsheet → Catalog → `randNorm(sd, mean, n)` where `n` is the number of samples. Show histogram with Histogram key in top left
- To calculate parameters of a dataset: Calc → One-variable

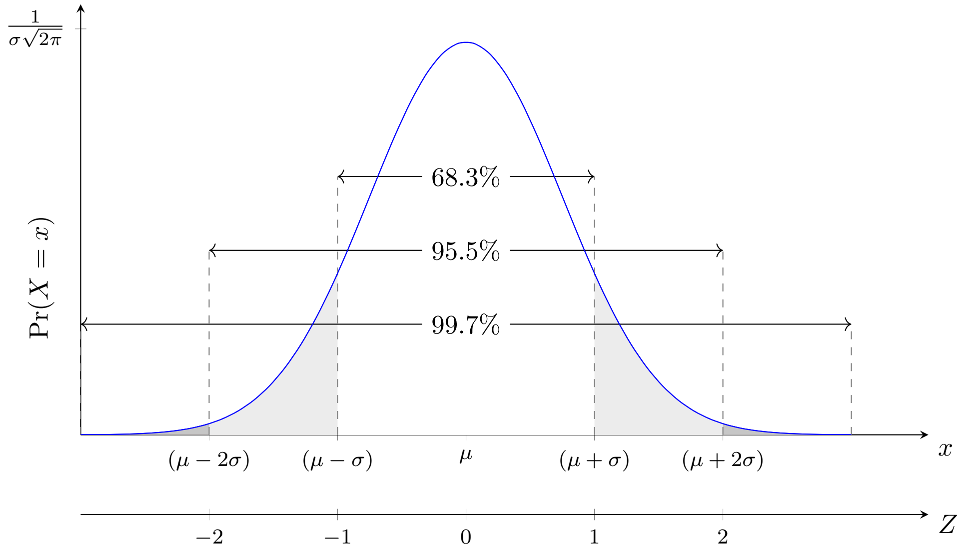
## Normal distributions

$$Z = \frac{X - \mu}{\sigma}$$

Normal distributions must have area (total prob.) of 1 $\implies \int_{-\infty}^{\infty} f(x)\,dx = 1$

mean = mode = median

**Always express $z$ as +ve. Express confidence interval as ordered pair.**




## Central limit theorem

If $X$ is randomly distributed with mean $\mu$ and sd $\sigma$, then with an adequate sample size $n$ the distribution of the sample mean $\overline{X}$ is approximately normal with mean $E(\overline{X})$ and $\text{sd}(\overline{X}) = \frac{\sigma}{\sqrt{n}}$.

## Confidence intervals

- **Point estimate:** single-valued estimate of the population mean from the value of the sample mean $\overline{x}$
- **Interval estimate:** confidence interval for population mean $\mu$
- $C\%$ confidence interval $\implies$ $C\%$ of samples will contain population mean $\mu$

### 95% confidence interval

For 95% c.i. of population mean $\mu$:

$$x \in \left(\overline{x} \pm 1.96\frac{\sigma}{\sqrt{n}}\right)$$

where:

$\overline{x}$ is the sample mean

$\sigma$ is the population sd

$n$ is the sample size from which $\overline{x}$ was calculated

**On CAS**

Menu → Stats → Calc → Interval

Set *Type = One-Sample Z Int*

and select *Variable*

### Margin of error

For 95% confidence interval of $\mu$:

$$M = 1.96 \times \frac{\sigma}{\sqrt{n}}$$
$$\implies n = \left(\frac{1.96\sigma}{M}\right)^2$$

Always round $n$ up to a whole number of samples.

### General case

For $C$% c.i. of population mean $\mu$:

$$x \in \left(\overline{x} \pm k\frac{\sigma}{\sqrt{n}}\right)$$

where $k$ is such that $\Pr(-k < Z < k) = \frac{C}{100}$

### Confidence interval for multiple trials

For a set of $n$ confidence intervals (samples), there is $0.95^n$ chance that all $n$ intervals contain the population mean $\mu$.

## 2 Hypothesis testing

**Note hypotheses are always expressed in terms of population parameters**

### Null hypothesis $H_0$

Sample drawn from population has same mean as control population, and any difference can be explained by sample variations.

### Alternative hypothesis $H_1$

Amount of variation from control is significant, despite standard sample variations.

### $p$-value

$$\begin{aligned} p &= \Pr(\overline{X} \lessgtr \mu(H_1)) \\ &= 2 \cdot \Pr(\overline{X} <> \mu(H_1) | \mu = 8) \end{aligned}$$

Probability of observing a value of the sample statistic as significant as the one observed, assuming null hypothesis is true.

| $p$ | Conclusion |
|---|---|
| $> 0.05$ | insufficient evidence against $H_0$ |
| $< 0.05$ (5%) | good evidence against $H_0$ |
| $< 0.01$ (1%) | strong evidence against $H_0$ |
| $< 0.001$ (0.1%) | very strong evidence against $H_0$ |

### Statistical significance

Significance level is denoted by $\alpha$.

If $p < \alpha$, null hypothesis is **rejected**

If $p > \alpha$, null hypothesis is **accepted**

### $z$-test

Hypothesis test for a mean of a sample drawn from a normally distributed population with a known standard deviation.

On CAS

Menu → Statistics → Calc → Test.

Select *One-Sample Z-Test* and *Variable*, then input:

| | |
|---|---|
| $\mu$ cond: | same operator as $H_1$ |
| $\mu_0$: | expected sample mean (null hypothesis) |
| $\sigma$: | standard deviation (null hypothesis) |
| $\overline{x}$: | sample mean |
| $n$: | sample size |

### One-tail and two-tail tests

#### One tail

- $\mu$ has changed in one direction
- State "$H_1 : \mu \lesseqgtr$ known population mean"

#### Two tail

- Direction of $\Delta\mu$ is ambiguous
- State "$H_1 : \mu \neq$ known population mean"

For two tail tests:

$$\begin{aligned} p\text{-value} &= \Pr(|\overline{X} - \mu| \geq |\overline{x}_0 - \mu|) \\ &= \left( |Z| \geq \left| \frac{\overline{x}_0 - \mu}{\sigma \div \sqrt{n}} \right| \right) \end{aligned}$$

### Modulus notation for two tail

$\Pr(|\overline{X} - \mu| \geq a) \implies$ "the probability that the distance between $\overline{\mu}$ and $\mu$ is $\geq a$"

### Inverse normal

**On CAS**

`invNormCdf("L",` $\alpha$, $\dfrac{\sigma}{n^{\alpha}}$, $\mu$`)`

### Errors

**Type I error** $H_0$ is rejected when it is **true**

**Type II error** $H_0$ is **not** rejected when it is **false**