Year 12 Methods

Andrew Lorimer

1 Functions

- vertical line test
- each x value produces only one y value

One to one functions

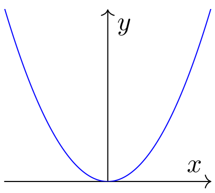
- $f(x)$ is *one to one* if $f(a) \neq f(b)$ if $a, b \in \text{dom}(f)$ and $a \neq b$
  $\implies$ unique y for each x ($\sin x$ is not 1:1, x^3 is)
- horizontal line test
- if not one to one, it is many to one

Odd and even functions

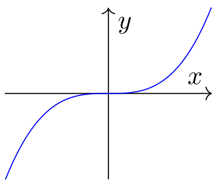
$$\text{Even:} \quad f(x) = f(-x)$$
$$\text{Odd:} \quad -f(x) = f(-x)$$

Even $\implies$ symmetrical across y-axis
$x^{\pm \frac{p}{q}}$ is odd if q is odd
For x^n, parity of $n \equiv$ parity of function

Even:



Odd:



Inverse functions

- Inverse of $f(x)$ is denoted $f^{-1}(x)$
- f must be one to one
- If $f(g(x)) = x$, then g is the inverse of f
- Represents reflection across $y = x$
- $\implies f^{-1}(x) = f(x)$ intersections lie on $y = x$
- ran f = dom f^{-1}
  dom f = ran f^{-1}
- "Inverse" $\neq$ "inverse *function*" (functions must pass vertical line test)

Finding f^{-1}

1. Let $y = f(x)$
2. Swap x and y ("take inverse"
3. Solve for y
   Sqrt: state $\pm$ solutions then restrict
4. State rule as $f^{-1}(x) = \dots$
5. For inverse *function*, state in function notation

Simultaneous equations (linear)

- **Unique solution -** lines intersect at point
- **Infinitely many solutions -** lines are equal
- **No solution -** lines are parallel

Solving $\begin{cases} px + qy = a \\ rx + sy = b \end{cases}$ **for** $\{0, 1, \infty\}$ **solutions**

where all coefficients are known except for one, and a, b are known

1. Write as matrices: $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$
2. Find determinant of first matrix: $\Delta = ps - qr$
3. Let $\Delta = 0$ for number of solutions $\neq 1$
   or let $\Delta \neq 0$ for one unique solution.
4. Solve determinant equation to find variable
   For infinite/no solutions:
5. Substitute variable into both original equations
6. Rearrange equations so that LHS of each is the same
7. $\text{RHS}(1) = \text{RHS}(2) \implies (1) = (2)\ \forall x$ (∞ solns)
   $\text{RHS}(1) \neq \text{RHS}(2) \implies (1) \neq (2)\ \forall x$ (0 solns)

On CAS: Matrix $\rightarrow$ `det`

Solving $\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$

- Use elimination

- Generate two new equations with only two variables
- Rearrange & solve
- Substitute one variable into another equation to find another variable

Piecewise functions

$$\text{e.g.} \quad f(x) = \begin{cases} x^{1/3}, & x \le 0 \\ 2, & 0 < x < 2 \\ x, & x \ge 2 \end{cases}$$

Open circle: point included

Closed circle: point not included

Operations on functions

For $f \pm g$ and $f \times g$: $\text{dom}' = \text{dom}(f) \cap \text{dom}(g)$

Addition of linear piecewise graphs: add y-values at key points

Product functions:

- product will equal 0 if $f = 0$ or $g = 0$
- $f'(x) = 0 \veebar g'(x) = 0 \not\Rightarrow (f \times g)'(x) = 0$

Composite functions

$(f \circ g)(x)$ is defined iff $\text{ran}(g) \subseteq \text{dom}(f)$

2 Polynomials

Linear equations

Forms

- $y = mx + c$
- $\frac{x}{a} + \frac{y}{b} = 1$ where (x_1, y_1) lies on the graph
- $y - y_1 = m(x - x_1)$ where $(a, 0)$ and $(0, b)$ are x- and y-intercepts

Line properties

Parallel lines: $m_1 = m_2$

Perpendicular lines: $m_1 \times m_2 = -1$

Distance: $|\vec{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Quadratics

$$x^2 + bx + c = (x + m)(x + n)$$

where $mn = c$, $m + n = b$

Difference of squares

$$a^2 - b^2 = (a - b)(a + b)$$

Perfect squares

$$a^2 \pm 2ab + b^2 = (a \pm b^2)$$

Completing the square

$$x^2 + bx + c = (x + \frac{b}{2})^2 + c - \frac{b^2}{4}$$

$$ax^2 + bx + c = a(x - \frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(Discriminant $\Delta = b^2 - 4ac$)

Cubics

Difference of cubes

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Sum of cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Perfect cubes

$$a^3 \pm 3a^2b + 3ab^2 \pm b^3 = (a \pm b)^3$$

$$y = a(bx - h)^3 + c$$

- $m = 0$ at *stationary point of inflection* (i.e. $(\frac{h}{b}, k)$)
- $y = (x-a)^2(x-b)$ — max at $x = a$, min at $x = b$
- $y = a(x - b)(x - c)(x - d)$ — roots at b, c, d
- $y = a(x-b)^2(x-c)$ — roots at b (instantaneous), c (intercept)

Quartic graphs

Forms of quartic equations

$y = ax^4$

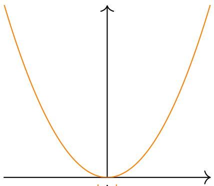
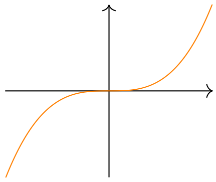
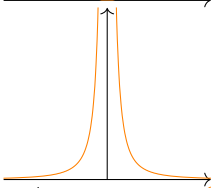
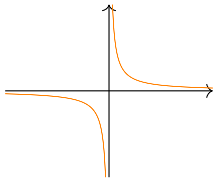
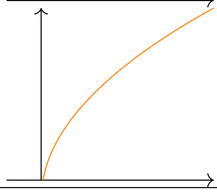
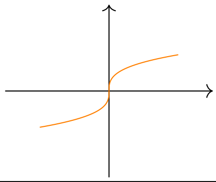
$y = a(x - b)(x - c)(x - d)(x - e)$

$y = ax^4 + cd^2 (c \ge 0)$

$y = ax^2(x - b)(x - c)$

$y = a(x - b)^2(x - c)^2$

$y = a(x - b)(x - c)^3$

	n is even	n is odd
$x^n, n \in \mathbb{Z}^+$		
$x^n, n \in \mathbb{Z}^-$		
$x^{\frac{1}{n}}, n \in \mathbb{Z}^-$		

3 Transformations

Order of operations: DRT

dilations — reflections — translations

Transforming x^n to $a(x-h)^n + K$

- dilation factor of $|a|$ units parallel to y-axis or from x-axis
- if $a < 0$, graph is reflected over x-axis
- translation of k units parallel to y-axis or from x-axis
- translation of h units parallel to x-axis or from y-axis
- for $(ax)^n$, dilation factor is $\frac{1}{a}$ parallel to x-axis or from y-axis
- when $0 < |a| < 1$, graph becomes closer to axis

Transforming $f(x)$ to $y = Af[n(x+c)] + b$

Applies to exponential, log, trig, e^x, polynomials.
Functions must be written in form $y = Af[n(x+c)] + b$

- dilation by factor $|A|$ from x-axis (if $A < 0$, reflection across y-axis)
- dilation by factor $\frac{1}{n}$ from y-axis (if $n < 0$, reflection across x-axis)
- translation of c units from y-axis (x-shift)
- translation of b units from x-axis (y-shift)

Dilations

Two pairs of equivalent processes for $y = f(x)$:

1. - Dilating from x-axis: $(x, y) \to (x, by)$
   - Replacing y with $\frac{y}{b}$ to obtain $y = bf(x)$
2. - Dilating from y-axis: $(x, y) \to (ax, y)$
   - Replacing x with $\frac{x}{a}$ to obtain $y = f(\frac{x}{a})$

For graph of $y = \frac{1}{x}$, horizontal & vertical dilations are equivalent (symmetrical). If $y = \frac{a}{x}$, graph is contracted rather than dilated.

Matrix transformations

Find new point (x', y'). Substitute these into original equation to find image with original variables (x, y).

Reflections

- Reflection **in** axis = reflection **over** axis = reflection **across** axis
- Translations do not change

Translations

For $y = f(x)$, these processes are equivalent:

- applying the translation $(x, y) \to (x + h, y + k)$ to the graph of $y = f(x)$

- replacing x with $x-h$ and y with $y-k$ to obtain $y-k=f(x-h)$

Power functions

Mostly only on CAS.

We can write $x^{\frac{-1}{n}} = \frac{1}{x^{\frac{1}{n}}} = \frac{1}{\sqrt[n]{x}}$n.

Domain is: $\begin{cases} \mathbb{R} \setminus \{0\} & \text{if } n \text{ is odd} \\ \mathbb{R}^+ & \text{if } n \text{ is even} \end{cases}$

If n is odd, it is an odd function.

$x^{\frac{p}{q}}$ where $p, q \in \mathbb{Z}^+$

$$x^{\frac{p}{q}} = \sqrt[q]{x^p}$$

- if $p > q$, the shape of x^p is dominant
- if $p < q$, the shape of $x^{\frac{1}{q}}$ is dominant
- points $(0,0)$ and $(1,1)$ will always lie on graph
- Domain is: $\begin{cases} \mathbb{R} & \text{if } q \text{ is odd} \\ \mathbb{R}^+ \cup \{0\} & \text{if } q \text{ is even} \end{cases}$

4 Exponentials & Logarithms

Logarithmic identities

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b x^n = n \log_b x$$

$$\log_b y^{x^n} = x^n \log_b y$$

$$\log_a(\frac{m}{n}) = \log_a m - \log_a$$

$$\log_a(m^{-1}) = -\log_a m$$

$$\log_b c = \frac{\log_a c}{\log_a b}$$

Index identities

$$b^{m+n} = b^m \cdot b^n$$

$$(b^m)^n = b^{m \cdot n}$$

$$(b \cdot c)^n = b^n \cdot c^n$$

$$b^m \div a^n = b^{m-n}$$

Inverse functions

For $f : \mathbb{R} \to \mathbb{R}, f(x) = a^x$, inverse is:

$$f^{-1} : \mathbb{R}^+ \to \mathbb{R}, f^{-1} = \log_a x$$

Euler's number e

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Modelling

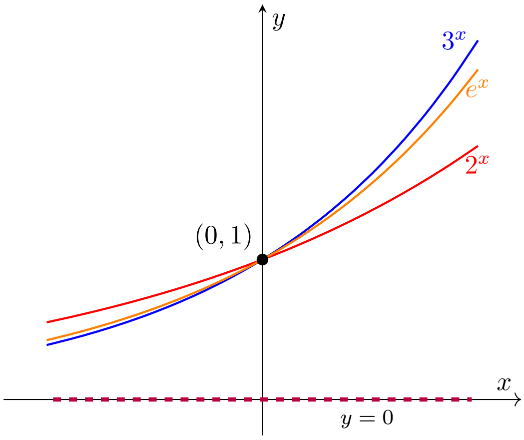
$$A = A_0 e^{kt}$$

- A_0 is initial value
- t is time taken
- k is a constant
- For continuous growth, $k > 0$
- For continuous decay, $k < 0$

Graphing exponential functions

$$f(x) = Aa^{k(x-b)} + c, \quad | \, a > 1$$

- **y-intercept** at $(0, A \cdot a^{-kb} + c)$ as $x \to \infty$
- **horizontal asymptote** at $y = c$
- **domain** is $\mathbb{R}$
- **range** is (c, ∞)
- dilation of factor $|A|$ from x-axis
- dilation of factor $\frac{1}{k}$ from y-axis



Graphing logarithmic functions

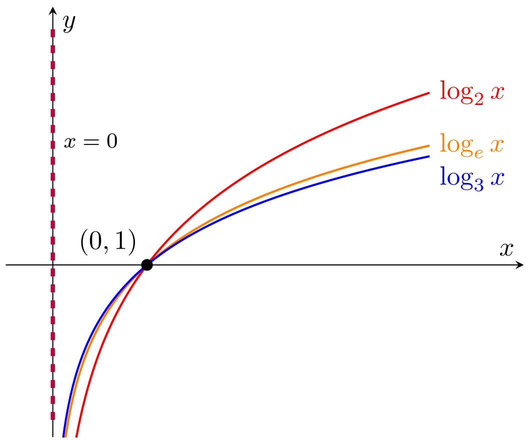
$\log_e x$ is the inverse of e^x (reflection across $y = x$)

$$f(x) = A \log_a k(x-b) + c$$

where

- **domain** is (b, ∞)
- **range** is $\mathbb{R}$
- **vertical asymptote** at $x = b$
- y-intercept exists if $b < 0$

- dilation of factor $|A|$ from x-axis
- dilation of factor $\frac{1}{k}$ from y-axis



Finding equations

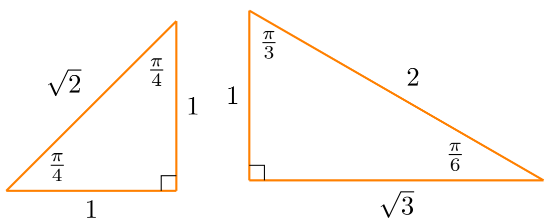
On CAS: $\begin{cases} f(3)=9 \\ g(3)=0 \end{cases} \Big|_{a,b}$

5 Circular functions

Radians and degrees

$$1 \text{ rad} = \frac{180 \deg}{\pi}$$

Exact values



Compound angle formulas

$$\cos(x \pm y) = \cos x + \cos y \mp \sin x \sin y$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

Double angle formulas

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \\ &= 2\cos^2 x - 1 \\ \sin 2x &= 2 \sin x \cos x \\ \tan 2x &= \frac{2\tan x}{1 - \tan^2 x} \end{aligned}$$

Symmetry

$$\sin(\theta + \frac{\pi}{2}) = \sin\theta$$

$$\sin(\theta + \pi) = -\sin\theta$$

$$\cos(\theta + \frac{\pi}{2}) = -\cos\theta$$

$$\begin{aligned} \cos(\theta + \pi) &= -\cos(\theta + \frac{3\pi}{2}) \\ &= \cos(-\theta) \end{aligned}$$

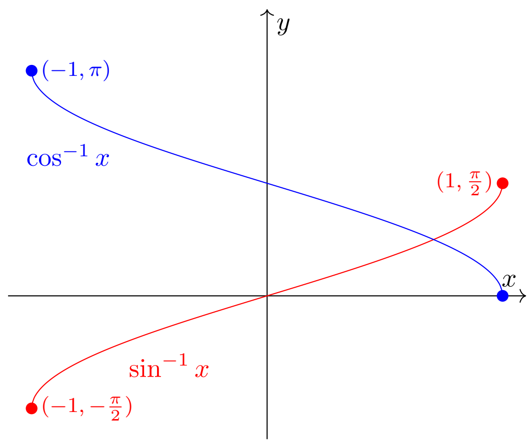
Complementary relationships

$$\begin{aligned} \sin\theta &= \cos(\frac{\pi}{2} - \theta) \\ &= -\cos(\theta + \frac{\pi}{2}) \\ \cos\theta &= \sin(\frac{\pi}{2} - \theta) \\ &= \sin(\theta + \frac{\pi}{2}) \end{aligned}$$

Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

Inverse circular functions



Inverse functions: $f(f^{-1}(x)) = x$ (restrict domain)

$$\sin^{-1} : [-1, 1] \to \mathbb{R}, \quad \sin^{-1} x = y$$

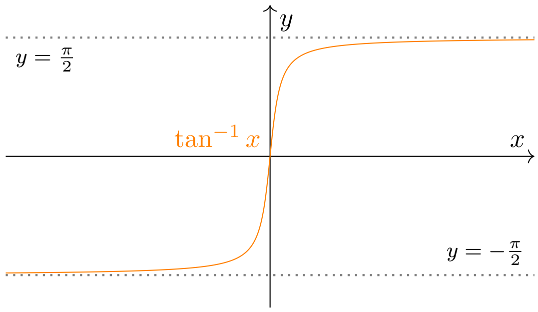
where $\sin y = x$, $y \in [\frac{-\pi}{2}, \frac{\pi}{2}]$

$$\cos^{-1} : [-1, 1] \to \mathbb{R}, \quad \cos^{-1} x = y$$

where $\cos y = x$, $y \in [0, \pi]$

$$\tan^{-1} : \mathbb{R} \to \mathbb{R}, \quad \tan^{-1} x = y$$

where $\tan y = x$, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



sin and cos graphs

$$f(x) = a \sin(bx - c) + d$$

where:

Period $= \frac{2\pi}{n}$

dom $= \mathbb{R}$

ran $= [-b + c, b + c]$;

$\cos(x)$ starts at $(0, 1)$, $\sin(x)$ starts at $(0, 0)$

0 amplitidue $\implies$ straight line

$a < 0$ or $b < 0$ inverts phase (swap sin and cos)

$c = T = \frac{2\pi}{b} \implies$ no net phase shift

tan graphs

$$y = a \tan(nx)$$

Period $= \dfrac{\pi}{n}$

Range is $\mathbb{R}$

Roots at $x = \dfrac{k\pi}{n}$ where $k \in \mathbb{Z}$

Asymptotes at $x = \frac{(2k+1)\pi}{2n}$

Asymptotes should always have equations

Solving trig equations

1. Solve domain for $n\theta$
2. Find solutions for $n\theta$
3. Divide solutions by n

$\sin 2\theta = \frac{\sqrt{3}}{2}, \quad \theta \in [0, 2\pi] \quad (\therefore 2\theta \in [0, 4\pi])$
$2\theta = \sin^{-1} \frac{\sqrt{3}}{2}$
$2\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$
$\therefore \theta = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$

6 Calculus

Average rate of change

$$m \text{ of } x \in [a, b] = \frac{f(b) - f(a)}{b - a} = \frac{dy}{dx}$$

On CAS: Action → Calculation → `diff`

Average value

$$f_{\text{avg}} = \frac{1}{b - a} \int_a^b f(x)\, dx$$

Instantaneous rate of change

Secant - line passing through two points on a curve

Chord - line segment joining two points on a curve

Limit theorems

1. For constant function $f(x) = k$, $\lim_{x \to a} f(x) = k$
2. $\lim_{x \to a}(f(x) \pm g(x)) = F \pm G$
3. $\lim_{x \to a}(f(x) \times g(x)) = F \times G$
4. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G}, G \neq 0$

A function is continuous if $L^- = L^+ = f(x)$ for all values of x.

First principles derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Not differentiable at:

- discontinuous points
- sharp point/cusp
- vertical tangents (∞ gradient)

Tangents & gradients

Tangent line - defined by $y = mx + c$ where $m = \frac{dy}{dx}$

Normal line - $\perp$ tangent ($m_{tan} \cdot m_{\text{norm}} = -1$)

Secant $= \frac{f(x+h)-f(x)}{h}$

On CAS:

Action → Calculation → Line → `tanLine` or `normal`

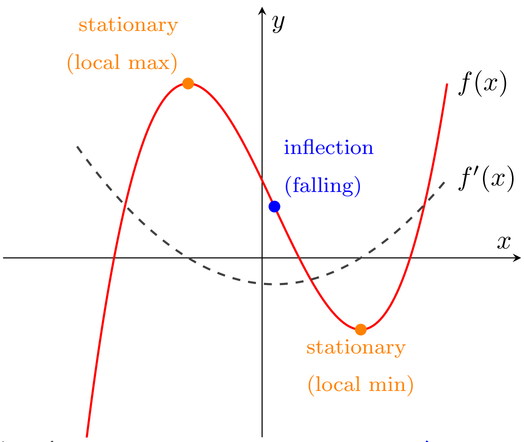
Solving on CAS

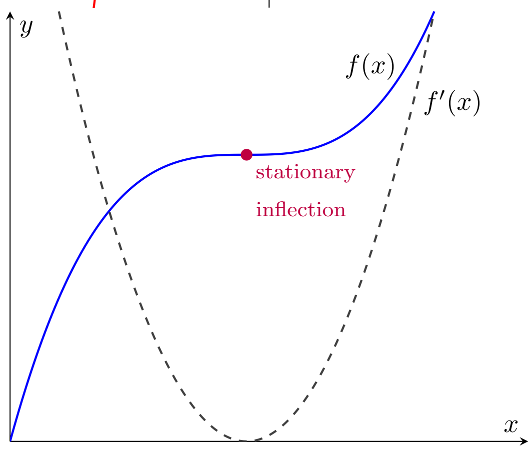
In main: type function. Interactive → Calculation → Line → (Normal | Tan line)

In graph: define function. Analysis → Sketch → (Normal | Tan line). Type x value to solve for a point. Return to show equation for line.

Stationary points

Stationary point:	$f'(x) = 0$
Point of inflection:	$f'' = 0$





Strictly increasing/decreasing

For x_2 and x_1 where $x_2 > x_1$:

- **strictly increasing**
  where $f(x_2) > f(x_1)$ or $f'(x) > 0$
- **strictly decreasing**
  where $f(x_2) < f(x_1)$ or $f'(x) < 0$
- Endpoints are included, even where gradient $= 0$

Derivatives

$f(x)$	$f'(x)$	
$\sin x$	$\cos x$	
$\sin ax$	$a \cos ax$	
$\cos x$	$-\sin x$	
$\cos ax$	$-a \sin ax$	
$\tan f(x)$	$f^2(x) \sec^2 f(x)$	
e^x	e^x	
e^{ax}	ae^{ax}	
ax^{nx}	$an \cdot e^{nx}$	
$\log_e x$	$\frac{1}{x}$	
$\log_e ax$	$\frac{1}{x}$	
$\log_e f(x)$	$\frac{f'(x)}{f(x)}$	
$\sin(f(x))$	$f'(x) \cdot \cos(f(x))$	
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$	
$\tan^{-1} x$	$\frac{1}{1+x^2}$	
$\frac{d}{dy} f(y)$	$\frac{1}{\frac{dx}{dy}}$	(reciprocal)
uv	$u\frac{dv}{dx} + v\frac{du}{dx}$	(product rule)
$\frac{u}{v}$	$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	(quotient rule)
$f(g(x))$	$f'(g(x)) \cdot g'(x)$	

Antiderivatives

$f(x)$	$\int f(x) \cdot dx$	
k (constant)	$kx + c$	
x^n	$\frac{1}{n+1} x^{n+1}$	
ax^{-n}	$a \cdot \log_e \lvert x \rvert + c$	
$\frac{1}{ax+b}$	$\frac{1}{a} \log_e(ax+b) + c$	
$(ax+b)^n$	$\frac{1}{a(n+1)}(ax+b)^{n-1} + c \mid n \neq 1$	
$(ax+b)^{-1}$	$\frac{1}{a} \log_e \lvert ax+b \rvert + c$	
e^{kx}	$\frac{1}{k} e^{kx} + c$	
e^k	$e^k x + c$	
$\sin kx$	$\frac{-1}{k} \cos(kx) + c$	
$\cos kx$	$\frac{1}{k} \sin(kx) + c$	
$\sec^2 kx$	$\frac{1}{k} \tan(kx) + c$	
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1} \frac{x}{a} + c \mid a > 0$	
$\frac{-1}{\sqrt{a^2-x^2}}$	$\cos^{-1} \frac{x}{a} + c \mid a > 0$	
$\frac{a}{a^2-x^2}$	$\tan^{-1} \frac{x}{a} + c$	
$\frac{f'(x)}{f(x)}$	$\log_e f(x) + c$	
$\int f(u) \cdot \frac{du}{dx} \cdot dx$	$\int f(u) \cdot du$	(substitution)
$f(x) \cdot g(x)$	$\int [f'(x) \cdot g(x)] dx + \int [g'(x) f(x)] dx$	