Year 12 Methods

Andrew Lorimer

1 Functions

- vertical line test
- each x value produces only one y value

One to one functions

- $f(x)$ is 1:1 if $f(a) \neq f(b) \; \forall \; \{a, b\} \in \text{dom}(f)$

  $\implies$ unique y for each x
- e.g. $\sin x$ is not 1:1, x^3 is
- horizontal line test
- if not one to one, it is many to one

Odd and even functions

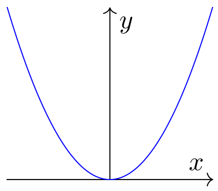
$$\begin{aligned} \text{Even:} \quad & f(x) = f(-x) \\ \text{Odd:} \quad & -f(x) = f(-x) \end{aligned}$$

Even $\implies$ symmetrical across y-axis

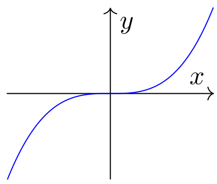
$x^{\pm \frac{p}{q}}$ is odd if q is odd

For x^n, parity of $n \equiv$ parity of function

Even:



Odd:



Inverse functions

- Inverse of $f(x)$ is denoted $f^{-1}(x)$
- f must be one to one
- If $f(g(x)) = x$, then g is the inverse of f
- Represents reflection across $y = x$
- $\implies f^{-1}(x) = f(x)$ intersections lie on $y = x$
- ran $f =$ dom f^{-1}

  dom $f =$ ran f^{-1}
- "Inverse" $\neq$ "inverse *function*" (functions must pass vertical line test)

Finding f^{-1}

1. Let $y = f(x)$
2. Swap x and y ("take inverse"
3. Solve for y

   Sqrt: state $\pm$ solutions then restrict
4. State rule as $f^{-1}(x) = \dots$
5. For inverse *function*, state in function notation

Simultaneous equations (linear)

- **Unique solution -** lines intersect at point
- **Infinitely many solutions -** lines are equal
- **No solution -** lines are parallel

$$\textbf{Solving} \begin{cases} px + qy = a \\ rx + sy = b \end{cases} \textbf{for } \{0, 1, \infty\} \textbf{ solutions}$$

where all coefficients are known except for one, and a, b are known

1. Write as matrices: $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$
2. Find det(first matrix) $= ps - qr$
3. Let det $= 0$ for $\{0, \infty\}$ solutions or det $\neq 0$ for 1 solution
4. Solve to find variable

For infinite/no solutions:

5. Substitute variable into both original equations
6. Rearrange so that LHS of each is the same
7. ∞ solns: RHS(1) = RHS(2) $\implies$ (1) = (2) $\forall x$

   0 solns: RHS(1) $\neq$ RHS(2) $\implies$ (1) $\neq$ (2) $\forall x$

On CAS

Action → Matrix → Calculation → `det`

Solving $\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$

- Use elimination
- Generate two new equations with only two variables
- Rearrange & solve
- Substitute one variable into another equation to find another variable

Piecewise functions

$$\text{e.g.} \quad f(x) = \begin{cases} x^{1/3}, & x \le 0 \\ 2, & 0 < x < 2 \\ x, & x \ge 2 \end{cases}$$

Open circle: point included

Closed circle: point not included

Operations on functions

For $f \pm g$ and $f \times g$: $\quad \text{dom}' = \text{dom}(f) \cap \text{dom}(g)$

Addition of linear piecewise graphs: add y-values at key points

Product functions:

- product will equal 0 if $f = 0$ or $g = 0$
- $f'(x) = 0 \veebar g'(x) = 0 \not\Rightarrow (f \times g)'(x) = 0$

Composite functions

$(f \circ g)(x)$ is defined iff $\text{ran}(g) \subseteq \text{dom}(f)$

2 Polynomials

Factor theorem

General form $\beta x + \alpha$

If $\beta x + \alpha$ is a factor of $P(x)$,

then $P(-\frac{\alpha}{\beta}) = 0$.

Simple form $x - a$

If $(x - a)$ is a factor of $P(x)$, remainder $R = 0$.

$\implies P(a) = 0$

Remainder theorem

When $P(x)$ is divided by $\beta x + \alpha$, the remainder is $-\frac{\alpha}{\beta}$.

Rational root theorem

Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial of degree n with $a_i \in \mathbb{Z} \forall a$. Let $\alpha, \beta \in \mathbb{Z}$ such that their highest common factor is 1 (i.e. relatively prime). If $\beta x + \alpha$ is a factor of $P(x)$, then β divides a_n and α divides a_0 .

Discriminant

$$\begin{cases} b^2 - 4ac > 0 & \text{two solutions} \\ b^2 - 4ac = 0 & \text{one solution} \\ b^2 - 4ac < 0 & \text{no solutions} \end{cases}$$

Flip inequality sign when multiplying by -1

Long division

$$\begin{array}{r} x + 3 \\ x - 1 \overline{) \quad x^2 + 2x + 4} \\ \underline{-\, x^2 \;\; + x \quad\;\;} \\ 3x + 4 \\ \underline{-\, 3x + 3} \\ 7 \end{array}$$

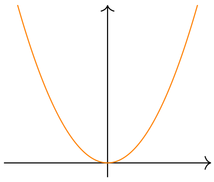
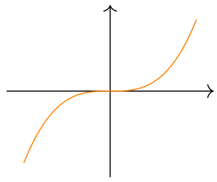
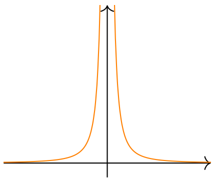
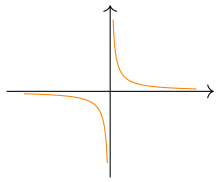
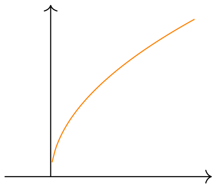
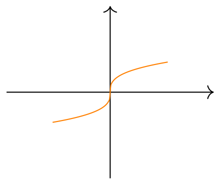
On CAS

Action → Transformation → `propFrac`

Linear equations

Forms

- $y = mx + c$
- $\frac{x}{a} + \frac{y}{b} = 1$ where (x_1, y_1) lies on the graph
- $y - y_1 = m(x - x_1)$ where $(a, 0)$ and $(0, b)$ are x- and y-intercepts

	n is even	n is odd
$x^n, n \in \mathbb{Z}^+$		
$x^n, n \in \mathbb{Z}^-$		
$x^{\frac{1}{n}}, n \in \mathbb{Z}^-$		

Line properties

Parallel lines: $m_1 = m_2$

Perpendicular lines: $m_1 \times m_2 = -1$

Distance: $|\vec{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Quadratics

Linear factorisation

$$x^2 + bx + c = (x + m)(x + n)$$

where $mn = c$, $m + n = b$

Difference of squares

$$a^2 - b^2 = (a - b)(a + b)$$

Perfect squares

$$a^2 \pm 2ab + b^2 = (a \pm b^2)$$

Completing the square

$$x^2 + bx + c = (x + \frac{b}{2})^2 + c - \frac{b^2}{4}$$

$$ax^2 + bx + c = a(x - \frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(Discriminant $\Delta = b^2 - 4ac$)

Cubics

Difference of cubes

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Sum of cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Perfect cubes

$$a^3 \pm 3a^2b + 3ab^2 \pm b^3 = (a \pm b)^3$$

$$y = a(bx - h)^3 + c$$

- $m = 0$ at *stationary point of inflection* (i.e. $(\frac{h}{b}, k)$)
- $y = (x - a)^2(x - b)$ — max at $x = a$, min at $x = b$
- $y = a(x - b)(x - c)(x - d)$ — roots at b, c, d
- $y = a(x - b)^2(x - c)$ — roots at b (instantaneous), c (intercept)

Quartic graphs

Forms of quartic equations

$y = ax^4$

$y = a(x - b)(x - c)(x - d)(x - e)$

$y = ax^4 + cd^2(c \geq 0)$

$y = ax^2(x - b)(x - c)$

$y = a(x - b)^2(x - c)^2$

$y = a(x - b)(x - c)^3$

3 Transformations

Order of operations: DRT

dilations — reflections — translations

Transforming x^n to $a(x-h)^n+K$

- dilation factor of $|a|$ units parallel to y-axis or from x-axis
- if $a<0$, graph is reflected over x-axis
- translation of k units $\parallel$ y-axis/from x-axis
- translation of h units $\parallel$ x-axis/from y-axis
- for $(ax)^n$, dilation factor is $\frac{1}{a \parallel x}$-axis/from y-axis
- when $0<|a|<1$, graph becomes closer to axis

Transforming $f(x)$ to $y = Af[n(x+c)]+b$

Applies to exponential, log, trig, e^x, polynomials.
Functions must be written in form $y = Af[n(x+c)]+b$

- dilation by factor $|A|$ from x-axis (if $A<0$, reflection across y-axis)
- dilation by factor $\frac{1}{n}$ from y-axis (if $n<0$, reflection across x-axis)
- translation of c units from y-axis (x-shift)
- translation of b units from x-axis (y-shift)

Dilations

Two pairs of equivalent processes for $y=f(x)$:

1. - Dilating from x-axis: $(x,y) \to (x, by)$
   - Replacing y with $\frac{y}{b}$ to obtain $y = bf(x)$
2. - Dilating from y-axis: $(x,y) \to (ax, y)$
   - Replacing x with $\frac{x}{a}$ to obtain $y = f(\frac{x}{a})$

For graph of $y=\frac{1}{x}$, horizontal & vertical dilations are equivalent (symmetrical). If $y=\frac{a}{x}$, graph is contracted rather than dilated.

Matrix transformations

Find new point (x', y'). Substitute these into original equation to find image with original variables (x,y).

Reflections

- Reflection **in** axis = reflection **over** axis = reflection **across** axis
- Translations do not change

Translations

For $y=f(x)$, these processes are equivalent:

- applying the translation $(x,y) \to (x+h, y+k)$ to the graph of $y=f(x)$
- replacing x with $x-h$ and y with $y-k$ to obtain $y-k=f(x-h)$

Power functions

Mostly only on CAS.
We can write $x^{\frac{-1}{n}} = \frac{1}{x^{\frac{1}{n}}} = \frac{1}{\sqrt[n]{x}}$n.

Domain is: $\begin{cases} \mathbb{R} \setminus \{0\} & \text{if } n \text{ is odd} \\ \mathbb{R}^+ & \text{if } n \text{ is even} \end{cases}$

If n is odd, it is an odd function.

$x^{\frac{p}{q}}$ where $p, q \in \mathbb{Z}^+$

$$x^{\frac{p}{q}} = \sqrt[q]{x^p}$$

- if $p>q$, the shape of x^p is dominant
- if $p<q$, the shape of $x^{\frac{1}{q}}$ is dominant
- points $(0,0)$ and $(1,1)$ will always lie on graph
- Domain is: $\begin{cases} \mathbb{R} & \text{if } q \text{ is odd} \\ \mathbb{R}^+ \cup \{0\} & \text{if } q \text{ is even} \end{cases}$

4 Exponentials & Logarithms

Logarithmic identities

$$\log_b(xy) = \log_b x + \log_b y$$
$$\log_b x^n = n \log_b x$$
$$\log_b y^{x^n} = x^n \log_b y$$
$$\log_a(\frac{m}{n}) = \log_a m - \log_a$$
$$\log_a(m^{-1}) = -\log_a m$$
$$\log_b c = \frac{\log_a c}{\log_a b}$$

Index identities

$$b^{m+n} = b^m \cdot b^n$$
$$(b^m)^n = b^{m \cdot n}$$
$$(b \cdot c)^n = b^n \cdot c^n$$
$$b^m \div a^n = b^{m-n}$$

Inverse functions

For $f : \mathbb{R} \to \mathbb{R}, f(x) = a^x$, inverse is:

$$f^{-1} : \mathbb{R}^+ \to \mathbb{R}, f^{-1} = \log_a x$$

Euler's number e

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Modelling

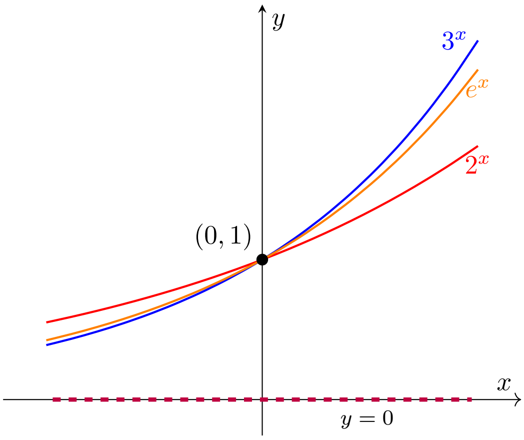
$$A = A_0 e^{kt}$$

- A_0 is initial value
- t is time taken
- k is a constant
- For continuous growth, $k > 0$
- For continuous decay, $k < 0$

Graphing exponential functions

$$f(x) = Aa^{k(x-b)} + c, \quad | a > 1$$

- **y-intercept** at $(0, A \cdot a^{-kb} + c)$ as $x \to \infty$
- **horizontal asymptote** at $y = c$
- **domain** is $\mathbb{R}$
- **range** is (c, ∞)
- dilation of factor $|A|$ from x-axis
- dilation of factor $\frac{1}{k}$ from y-axis



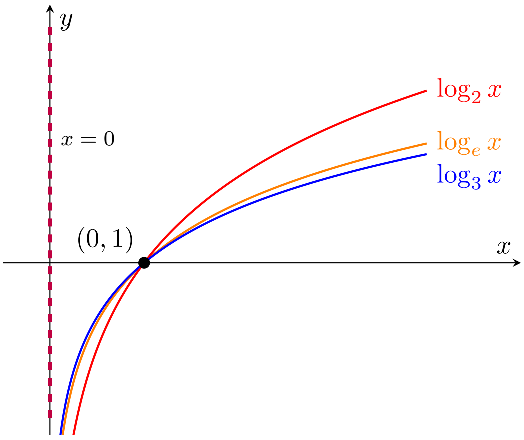
Graphing logarithmic functions

$\log_e x$ is the inverse of e^x (reflection across $y = x$)

$$f(x) = A \log_a k(x - b) + c$$

where

- **domain** is (b, ∞)
- **range** is $\mathbb{R}$
- **vertical asymptote** at $x = b$
- y-intercept exists if $b < 0$
- dilation of factor $|A|$ from x-axis
- dilation of factor $\frac{1}{k}$ from y-axis



Finding equations

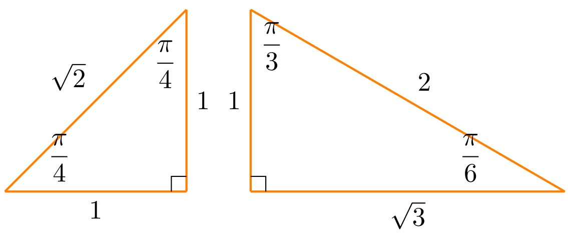
On CAS: $\begin{cases} f(3)=9 \\ g(3)=0 \end{cases} \Big|_{a,b}$

5 Circular functions

Radians and degrees

$$1 \text{ rad} = \frac{180 \deg}{\pi}$$

Exact values



Compound angle formulas

$$\cos(x \pm y) = \cos x + \cos y \mp \sin x \sin y$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

Double angle formulas

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 1 - 2\sin^2 x$$
$$= 2\cos^2 x - 1$$
$$\sin 2x = 2\sin x \cos x$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Symmetry

$$\sin(\theta + \frac{\pi}{2}) = \sin\theta$$
$$\sin(\theta + \pi) = -\sin\theta$$

$$\cos(\theta + \frac{\pi}{2}) = -\cos\theta$$
$$\cos(\theta + \pi) = -\cos(\theta + \frac{3\pi}{2})$$
$$= \cos(-\theta)$$

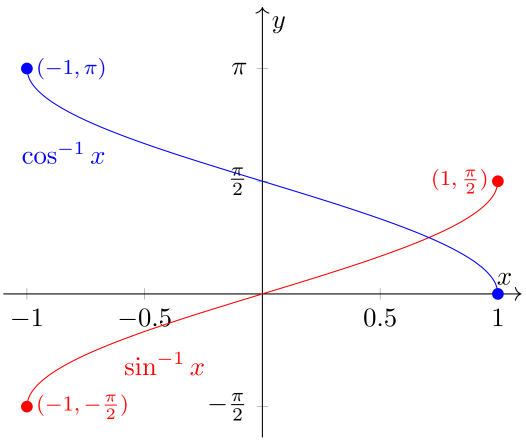
Complementary relationships

$$\sin\theta = \cos(\frac{\pi}{2} - \theta)$$
$$= -\cos(\theta + \frac{\pi}{2})$$
$$\cos\theta = \sin(\frac{\pi}{2} - \theta)$$
$$= \sin(\theta + \frac{\pi}{2})$$

Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

Inverse circular functions

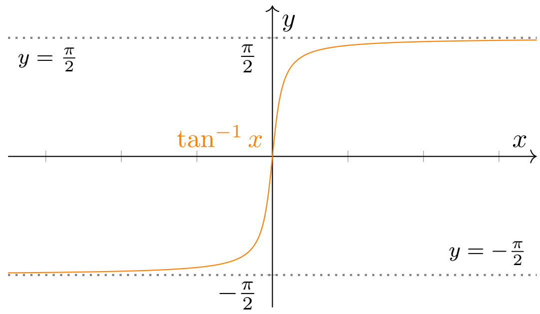


Inverse functions: $f(f^{-1}(x)) = x$ (restrict domain)

$$\sin^{-1} : [-1, 1] \to \mathbb{R}, \quad \sin^{-1} x = y$$
where $\sin y = x, \ y \in [\frac{-\pi}{2}, \frac{\pi}{2}]$

$$\cos^{-1} : [-1, 1] \to \mathbb{R}, \quad \cos^{-1} x = y$$
where $\cos y = x, \ y \in [0, \pi]$

$$\tan^{-1} : \mathbb{R} \to \mathbb{R}, \quad \tan^{-1} x = y$$
where $\tan y = x, \ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$\sin$ and $\cos$ graphs

$$f(x) = a\sin(bx - c) + d$$

where:

Period $= \frac{2\pi}{n}$

dom $= \mathbb{R}$

ran $= [-b + c, b + c]$;

$\cos(x)$ starts at $(0, 1)$, $\sin(x)$ starts at $(0, 0)$

0 amplitidue $\implies$ straight line

$a < 0$ or $b < 0$ inverts phase (swap sin and cos)

$c = T = \frac{2\pi}{b} \implies$ no net phase shift

$\tan$ graphs

$$y = a\tan(nx)$$

Period $= \dfrac{\pi}{n}$

Range is $\mathbb{R}$

Roots at $x = \dfrac{k\pi}{n}$ where $k \in \mathbb{Z}$

Asymptotes at $x = \frac{(2k+1)\pi}{2n}$

Asymptotes should always have equations

Solving trig equations

1. Solve domain for $n\theta$
2. Find solutions for $n\theta$
3. Divide solutions by n

$\sin 2\theta = \frac{\sqrt{3}}{2}, \quad \theta \in [0, 2\pi] \quad (\therefore 2\theta \in [0, 4\pi])$
$2\theta = \sin^{-1} \frac{\sqrt{3}}{2}$
$2\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$
$\therefore \theta = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$

6 Calculus

Average rate of change

$$m \text{ of } x \in [a, b] = \frac{f(b) - f(a)}{b - a} = \frac{dy}{dx}$$

On CAS: Action → Calculation → `diff`

Average value

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

Instantaneous rate of change

Secant - line passing through two points on a curve

Chord - line segment joining two points on a curve

Limit theorems

1. For constant function $f(x) = k$, $\lim_{x \to a} f(x) = k$
2. $\lim_{x \to a}(f(x) \pm g(x)) = F \pm G$
3. $\lim_{x \to a}(f(x) \times g(x)) = F \times G$
4. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G}, G \neq 0$

A function is continuous if $L^- = L^+ = f(x)$ for all values of x.

First principles derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Not differentiable at:

- discontinuous points
- sharp point/cusp
- vertical tangents (∞ gradient)

Tangents & gradients

Tangent line - defined by $y = mx + c$ where $m = \frac{dy}{dx}$

Normal line - $\perp$ tangent ($m_{tan} \cdot m_{\text{norm}} = -1$)

Secant $= \frac{f(x+h)-f(x)}{h}$

On CAS

In main: Interactive → Calculation → Line

```
tanLine(f(x), x, p)
normal(f(x), x, p)
```

where `p` is the x-value of the coordinate

In graph: define function, then Analysis → Sketch → (Normal | Tan line). Type x value to solve for a point. Return to show equation for line.

Strictly increasing/decreasing

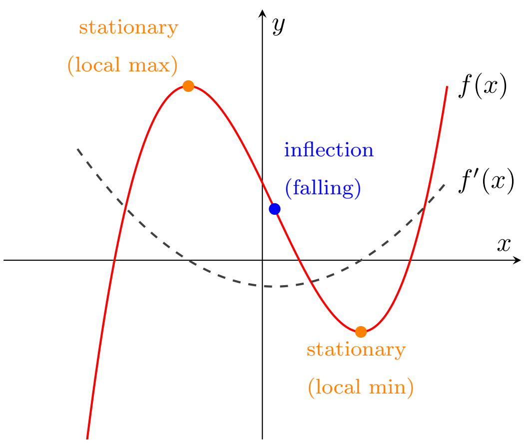
For x_2 and x_1 where $x_2 > x_1$:

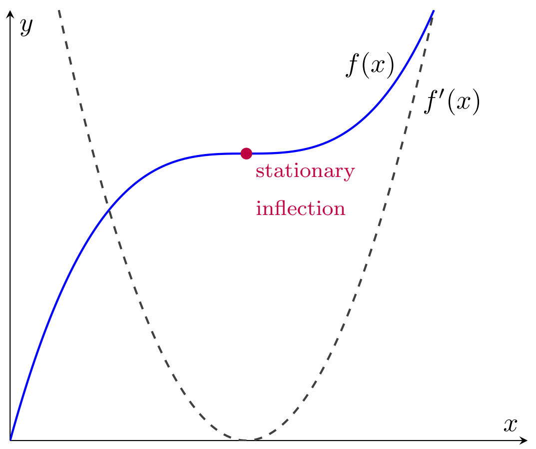
- **strictly increasing**
  where $f(x_2) > f(x_1)$ or $f'(x) > 0$
- **strictly decreasing**
  where $f(x_2) < f(x_1)$ or $f'(x) < 0$
- Endpoints are included, even where gradient $= 0$

Stationary points

Stationary point: $f'(x) = 0$

Point of inflection: $f'' = 0$





Derivatives

$f(x)$	$f'(x)$	
$\sin x$	$\cos x$	
$\sin ax$	$a \cos ax$	
$\cos x$	$-\sin x$	
$\cos ax$	$-a \sin ax$	
$\tan f(x)$	$f^2(x) \sec^2 f(x)$	
e^x	e^x	
e^{ax}	ae^{ax}	
ax^{nx}	$an \cdot e^{nx}$	
$\log_e x$	$\frac{1}{x}$	
$\log_e ax$	$\frac{1}{x}$	
$\log_e f(x)$	$\frac{f'(x)}{f(x)}$	
$\sin(f(x))$	$f'(x) \cdot \cos(f(x))$	
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$	
$\tan^{-1} x$	$\frac{1}{1+x^2}$	
$\frac{d}{dy} f(y)$	$\frac{1}{\frac{dx}{dy}}$	(reciprocal)
uv	$u\frac{dv}{dx} + v\frac{du}{dx}$	(product rule)
$\frac{u}{v}$	$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	(quotient rule)
$f(g(x))$	$f'(g(x)) \cdot g'(x)$	

Antiderivatives

$f(x)$	$\int f(x) \cdot dx$	
k (constant)	$kx + c$	
x^n	$\frac{1}{n+1}x^{n+1}$	
ax^{-n}	$a \cdot \log_e \lvert x \rvert + c$	
$\frac{1}{ax+b}$	$\frac{1}{a}\log_e(ax+b) + c$	
$(ax+b)^n$	$\frac{1}{a(n+1)}(ax+b)^{n-1} + c \mid n \neq 1$	
$(ax+b)^{-1}$	$\frac{1}{a}\log_e \lvert ax+b \rvert + c$	
e^{kx}	$\frac{1}{k}e^{kx} + c$	
e^k	$e^k x + c$	
$\sin kx$	$\frac{-1}{k}\cos(kx) + c$	
$\cos kx$	$\frac{1}{k}\sin(kx) + c$	
$\sec^2 kx$	$\frac{1}{k}\tan(kx) + c$	
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a} + c \mid a > 0$	
$\frac{-1}{\sqrt{a^2-x^2}}$	$\cos^{-1}\frac{x}{a} + c \mid a > 0$	
$\frac{a}{a^2-x^2}$	$\tan^{-1}\frac{x}{a} + c$	
$\frac{f'(x)}{f(x)}$	$\log_e f(x) + c$	
$\int f(u) \cdot \frac{du}{dx} \cdot dx$	$\int f(u) \cdot du$	(substitution)
$f(x) \cdot g(x)$	$\int [f'(x) \cdot g(x)]dx + \int [g'(x) f(x)]dx$	

7 Statistics

Probability

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

$$\Pr(A \cap B) = \Pr(A|B) \times \Pr(B)$$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$\Pr(A) = \Pr(A|B) \cdot \Pr(B) + \Pr(A|B') \cdot \Pr(B')$$

Mutually exclusive: $\Pr(A \cap B) = 0$

Independent events:

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$
$$\Pr(A|B) = \Pr(A)$$
$$\Pr(B|A) = \Pr(B)$$

Combinatorics

Arrangements $\binom{n}{k} = \frac{n!}{(n-k)}$

Combinations $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

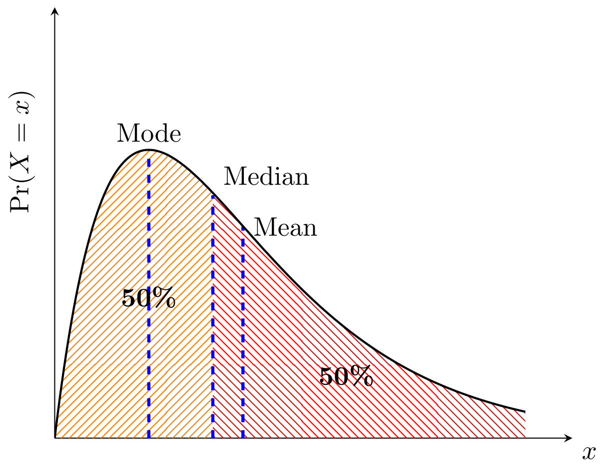
Note $\binom{n}{k} = \binom{n}{k-1}$

On CAS

Keyboard → Advance → nCr / nPr

`nCr(n, r)` or `nPr(n, r)`

Distributions



Mean μ

$$E(X) = \frac{\Sigma\left[x \cdot f(x)\right]}{\Sigma f} \qquad (f = \text{absolute frequency})$$
$$= \sum_{i=1}^{n} \left[x_i \cdot \Pr(X = x_i)\right] \qquad \text{(discrete)}$$
$$= \int_{\mathbf{X}} (x \cdot f(x))\, dx$$

Mode

Value of X which has the highest probability

- Most popular value in discrete distributions
- Must exist in distribution
- Represented by local max in pdf
- Multiple modes exist when $> 1 X$ value have equal-highest probability

Median

Value separating lower and upper half of distribution area

Continuous:

$$m = X \text{ such that } \int_{-\infty}^{m} f(x)\, dx = 0.5$$

Discrete: (not in course)

- Does not have to exist in distribution
- Add values of X smallest to largest until sum is ≥ 0.5
- If $X_1 < 0.5 < X_2$, then median is the average of X_1 and X_2
  - If $m > 0.5$, then value of X that is reached is the median of X

Variance σ^2

$$\begin{aligned} \text{Var}(x) &= \sum_{i=1}^{n} p_i (x_i - \mu)^2 \\ &= \sum (x-\mu)^2 \times \Pr(X = x) \\ &= \sum x^2 \times p(x) - \mu^2 \\ &= \text{E}(X^2) - [\text{E}(X)]^2 \\ &= E\left[(X-\mu)^2\right] \end{aligned}$$

Standard deviation σ

$$\begin{aligned} \sigma &= \text{sd}(X) \\ &= \sqrt{\text{Var}(X)} \end{aligned}$$

Binomial distributions

Conditions for a *binomial distribution*:

1. Two possible outcomes: **success** or **failure**
2. Pr(success) (=p) is constant across trials
3. Finite number n of independent trials

Properties of $X \sim \text{Bi}(n, p)$

$$\mu(X) = np$$
$$\text{Var}(X) = np(1-p)$$
$$\sigma(X) = \sqrt{np(1-p)}$$
$$\Pr(X = x) = \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x}$$

On CAS

Interactive → Distribution → `binomialPdf`

x:	no. of successes
numtrial:	no. of trials
pos:	probability of success

Continuous random variables

A continuous random variable X has a pdf f such that:

1. $f(x) \geq 0 \forall x$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$E(X) = \int_{\mathbf{X}} (x \cdot f(x))\,dx$$
$$\text{Var}(X) = E\left[(X-\mu)^2\right]$$

$$\Pr(X \leq c) = \int_{-\infty}^{c} f(x)\,dx$$

On CAS

Define piecewise functions:

Math3 → 

Two random variables X, Y

If X and Y are independent:

$$\text{E}(aX + bY) = a\,\text{E}(X) + b\,\text{E}(Y)$$
$$\text{Var}(aX \pm bY \pm c) = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y)$$

Linear functions $X \to aX + b$

$$\begin{aligned}\Pr(Y \leq y) &= \Pr(aX + b \leq y)\\ &= \Pr\left(X \leq \frac{y-b}{a}\right)\\ &= \int_{-\infty}^{\frac{y-b}{a}} f(x)\,dx\end{aligned}$$

Mean: $\text{E}(aX + b) = a\,\text{E}(X) + b$

Variance: $\text{Var}(aX + b) = a^2\,\text{Var}(X)$

Expectation theorems

For some non-linear function g, the expected value $E(g(X))$ is not equal to $g(E(X))$.

$$\begin{aligned}E(X^2) &= \text{Var}(X) - [E(X)]^2 &\\ E(X^n) &= \Sigma x^n \cdot p(x) & \text{(non-linear)}\\ &\neq [E(X)]^n &\\ E(aX \pm b) &= aE(X) \pm b & \text{(linear)}\\ E(b) &= b & (\forall b \in \mathbb{R})\\ E(X+Y) &= E(X) + E(Y) & \text{(two variables)}\end{aligned}$$

Sample mean

Approximation of the **population mean** determined experimentally.

$$\overline{x} = \frac{\Sigma x}{n}$$

where

n is the size of the sample (number of sample points)

x is the value of a sample point

On CAS

1. Spreadsheet
2. In cell A1:
   `mean(randNorm(sd, mean, sample size))`
3. Edit → Fill → Fill Range
4. Input range as A1:An where n is the number of samples
5. Graph → Histogram

Sample size of n

$$\overline{X} = \sum_{i=1}^{n} \frac{x_i}{n} = \frac{\sum x}{n}$$

Sample mean is distributed with mean μ and sd $\frac{\sigma}{\sqrt{n}}$ (approaches these values for increasing sample size n).

For a new distribution with mean of n trials, $\text{E}(X') = \text{E}(X), \quad \text{sd}(X') = \dfrac{\text{sd}(X)}{\sqrt{n}}$

On CAS

- Spreadsheet → Catalog → `randNorm(sd, mean, n)` where `n` is the number of samples. Show histogram with Histogram key in top left
- To calculate parameters of a dataset: Calc → One-variable

Population sampling

Population proportion

$$p = \frac{n \text{ with attribute in population}}{\text{population size}}$$

Constant for a given population.

Sample proportion

$$\hat{p} = \frac{n \text{ with attribute in sample}}{\text{sample size}}$$

Varies with each sample.

Normal distributions

$$Z = \frac{X - \mu}{\sigma}$$

Normal distributions must have area (total prob.) of 1 $\implies \int_{-\infty}^{\infty} f(x)\, dx = 1$

mean = mode = median

Always express z as +ve. Express confidence interval as ordered pair.

Confidence intervals

- **Point estimate:** single-valued estimate of the population mean from the value of the sample mean $\overline{x}$
- **Interval estimate:** confidence interval for population mean μ
- $C\%$ confidence interval $\implies$ $C\%$ of samples will contain population mean μ

On CAS

Menu → Stats → Calc → Interval

Set *Type* = *One-Sample Z Int*

and select *Variable*

95% confidence interval

For 95% c.i. of population mean μ:

$$x \in \left(\overline{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}\right)$$

where:

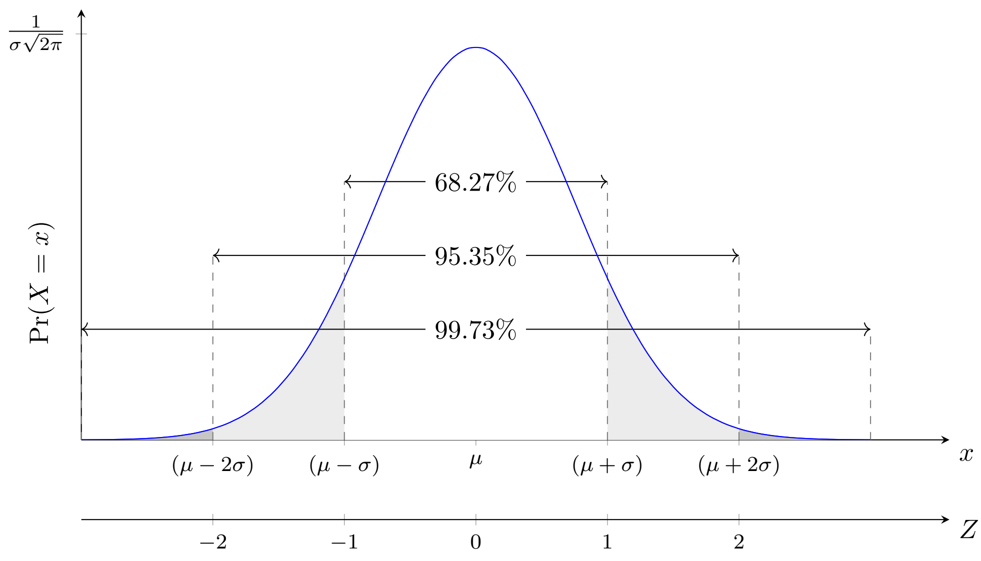
$\overline{x}$ is the sample mean

σ is the population sd

n is the sample size from which $\overline{x}$ was calculated

Confidence interval of p from $\hat{p}$

$$x \in \left(\hat{p} \pm Z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$



Margin of error

For 95% confidence interval of μ:

$$\begin{aligned} M &= 1.96 \times \frac{\sigma}{\sqrt{n}} \\ &= \frac{1}{2} \times \text{width of c.i.} \\ \implies n &= \left(\frac{1.96\sigma}{M}\right)^2 \end{aligned}$$

Always round n up to a whole number of samples.

General case

For C% c.i. of population mean μ:

$$x \in \left(\overline{x} \pm k\frac{\sigma}{\sqrt{n}}\right)$$

where k is such that $\Pr(-k < Z < k) = \frac{C}{100}$

On CAS

Menu → Stats → Calc → Interval

Set *Type* = *One-Prop Z Int*

Input x $= \hat{p} * n$

Confidence interval for multiple trials

For a set of n confidence intervals (samples), there is 0.95^n chance that all n intervals contain the population mean μ.