# 1 Complex numbers

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

Cartesian form: $a + bi$

Polar form: $r \operatorname{cis} \theta$

## Operations

| | **Cartesian** | **Polar** |
|---|---|---|
| $z_1 \pm z_2$ | $(a \pm c)(b \pm d)i$ | convert to $a + bi$ |
| $+k \times z$ | $ka \pm kbi$ | $kr \operatorname{cis} \theta$ |
| $-k \times z$ | | $kr \operatorname{cis}(\theta \pm \pi)$ |
| $z_1 \cdot z_2$ | $ac - bd + (ad + bc)i$ | $r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$ |
| $z_1 \div z_2$ | $(z_1 \overline{z_2}) \div \lvert z_2 \rvert^2$ | $\left(\frac{r_1}{r_2}\right) \operatorname{cis}(\theta_1 - \theta_2)$ |

### Scalar multiplication in polar form

For $k \in \mathbb{R}^+$:

$$k\,(r \operatorname{cis} \theta) = kr \operatorname{cis} \theta$$

For $k \in \mathbb{R}^-$:

$$k\,(r \operatorname{cis} \theta) = kr \operatorname{cis}\left(\begin{cases} \theta - \pi & |0 < \operatorname{Arg}(z) \le \pi \\ \theta + \pi & | -\pi < \operatorname{Arg}(z) \le 0 \end{cases}\right)$$

## Conjugate

$$\begin{aligned} \overline{z} &= a \mp bi \\ &= r \operatorname{cis}(-\theta) \end{aligned}$$

On CAS: `conjg(a+bi)`

### Properties

$$\begin{aligned} \overline{z_1 \pm z_2} &= \overline{z_1} \pm \overline{z_2} \\ \overline{z_1 \cdot z_2} &= \overline{z_1} \cdot \overline{z_2} \\ \overline{kz} &= k\overline{z} \quad | \quad k \in \mathbb{R} \\ z\overline{z} &= (a + bi)(a - bi) \\ &= a^2 + b^2 \\ &= |z|^2 \end{aligned}$$

## Modulus

$$|z| = |\vec{Oz}| = \sqrt{a^2 + b^2}$$

### Properties

$$\begin{aligned} |z_1 z_2| &= |z_1||z_2| \\ \left|\frac{z_1}{z_2}\right| &= \frac{|z_1|}{|z_2|} \\ |z_1 + z_2| &\le |z_1| + |z_2| \end{aligned}$$

## Multiplicative inverse

$$\begin{aligned} z^{-1} &= \frac{a - bi}{a^2 + b^2} \\ &= \frac{\overline{z}}{|z|^2} a \\ &= r \operatorname{cis}(-\theta) \end{aligned}$$

## Dividing over $\mathbb{C}$

$$\begin{aligned} \frac{z_1}{z_2} &= z_1 z_2^{-1} \\ &= \frac{z_1 \overline{z_2}}{|z_2|^2} \\ &= \frac{(a + bi)(c - di)}{c^2 + d^2} \end{aligned}$$

(rationalise denominator)

## Polar form

$$\begin{aligned} z &= r \operatorname{cis} \theta \\ &= r(\cos\theta + i \sin\theta) \end{aligned}$$

- $r = |z| = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}$
- $\theta = \arg(z)$ On CAS: `arg(a+bi)`
- $\operatorname{Arg}(z) \in (-\pi, \pi)$ **(principal argument)**
- Convert on CAS:
  `compToTrig(a+bi)` $\iff$ `cExpand{r·cisX}`
- Multiple representations:
  $r \operatorname{cis} \theta = r \operatorname{cis}(\theta + 2n\pi)$ with $n \in \mathbb{Z}$ revolutions
- $\operatorname{cis} \pi = -1, \qquad \operatorname{cis} 0 = 1$

## de Moivres' theorem

$$(r \operatorname{cis} \theta)^n = r^n \operatorname{cis}(n\theta) \text{ where } n \in \mathbb{Z}$$

## Complex polynomials

Include $\pm$ for all solutions, incl. imaginary

| | |
|---|---|
| Sum of squares | $z^2 + a^2 = z^2 - (ai)^2$ $= (z + ai)(z - ai)$ |
| Sum of cubes | $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$ |
| Division | $P(z) = D(z)Q(z) + R(z)$ |
| Remainder theorem | Let $\alpha \in \mathbb{C}$. Remainder of $P(z) \div (z - \alpha)$ is $P(\alpha)$ |
| Factor theorem | $z - \alpha$ is a factor of $P(z) \iff P(\alpha) = 0$ for $\alpha \in \mathbb{C}$ |
| Conjugate root theorem | $P(z) = 0$ at $z = a \pm bi$ ($\implies$ both $z_1$ and $\overline{z_1}$ are solutions) |

## Roots

$n$th roots of $z = r \operatorname{cis}\theta$ are:

$$z = r^{\frac{1}{n}} \operatorname{cis}\left(\frac{\theta + 2k\pi}{n}\right)$$

- Same modulus for all solutions
- Arguments are separated by $\frac{2\pi}{n}$
- Solutions of $z^n = a$ where $a \in \mathbb{C}$ lie on the circle $x^2 + y^2 = \left(|a|^{\frac{1}{n}}\right)^2$ (intervals of $\frac{2\pi}{n}$)

For $0 = az^2 + bz + c$, use quadratic formula:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

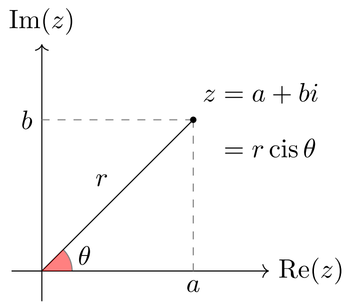
## Fundamental theorem of algebra

A polynomial of degree $n$ can be factorised into $n$ linear factors in $\mathbb{C}$:

$$\implies P(z) = a_n(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)\dots(z - \alpha_n)$$

$$\text{where } \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n \in \mathbb{C}$$

## Argand planes




- Multiplication by $i \implies$ CCW rotation of $\frac{\pi}{2}$
- Addition: $z_1 + z_2 \equiv \overrightarrow{Oz_1} + \overrightarrow{Oz_2}$

## Sketching complex graphs

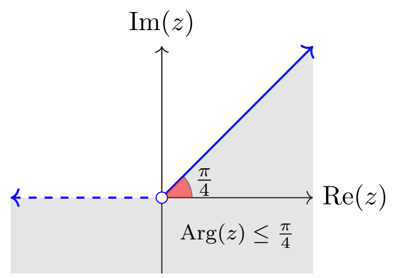
### Linear

- $\operatorname{Re}(z) = c$ or $\operatorname{Im}(z) = c$ (perpendicular bisector)
- $\operatorname{Im}(z) = m\operatorname{Re}(z)$
- $|z + a| = |z + b| \implies 2(a - b)x = b^2 - a^2$

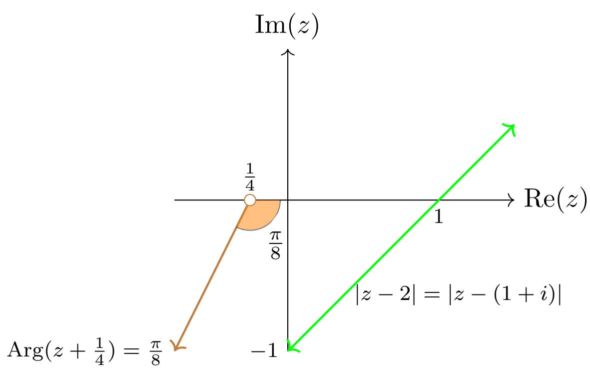
### Circles

- $|z - z_1|^2 = c^2|z_2 + 2|^2$
- $|z - (a + bi)| = c$

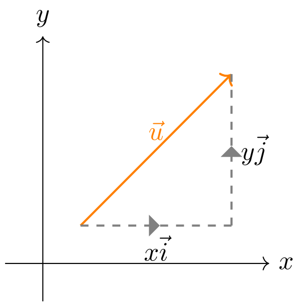
**Loci** $\operatorname{Arg}(z) < \theta$




**Rays** $\operatorname{Arg}(z - b) = \theta$



# 2 Vectors




## Column notation

$$\begin{bmatrix} x \\ y \end{bmatrix} \iff x\boldsymbol{i} + y\boldsymbol{j}$$

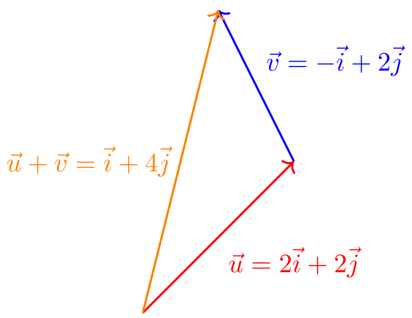
$$\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} \quad \text{between } A(x_1, y_1),\ B(x_2, y_2)$$

## Scalar multiplication

$$k \cdot (x\boldsymbol{i} + y\boldsymbol{j}) = kx\boldsymbol{i} + ky\boldsymbol{j}$$

For $k \in \mathbb{R}^-$, direction is reversed

## Vector addition




$$(x\boldsymbol{i} + y\boldsymbol{j}) \pm (a\boldsymbol{i} + b\boldsymbol{j}) = (x \pm a)\boldsymbol{i} + (y \pm b)\boldsymbol{j}$$

- Draw each vector head to tail then join lines
- Addition is commutative (parallelogram)
- $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$

## Magnitude

$$|(x\boldsymbol{i} + y\boldsymbol{j})| = \sqrt{x^2 + y^2}$$

## Parallel vectors

$$\boldsymbol{u}||\boldsymbol{v} \iff \boldsymbol{u} = k\boldsymbol{v} \text{ where } k \in \mathbb{R} \setminus \{0\}$$

For parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{cases} |\boldsymbol{a}||\boldsymbol{b}| & \text{if same direction} \\ -|\boldsymbol{a}||\boldsymbol{b}| & \text{if opposite directions} \end{cases}$$

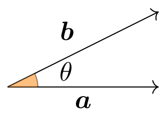
## Perpendicular vectors

$$\boldsymbol{a} \perp \boldsymbol{b} \iff \boldsymbol{a} \cdot \boldsymbol{b} = 0 \quad (\text{since } \cos 90 = 0)$$

## Unit vector $|\hat{\boldsymbol{a}}| = 1$

$$\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$$
$$= \boldsymbol{a} \cdot |\boldsymbol{a}|$$

## Scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$




$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$
$$= |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$$
$$(\ 0 \le \theta \le \pi) \text{ - from cosine rule}$$

On CAS: `dotP([a b c], [d e f])`

### Properties

1. $k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (k\boldsymbol{b})$
2. $\boldsymbol{a} \cdot \boldsymbol{0} = 0$
3. $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
4. $\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1$
5. $\boldsymbol{a} \cdot \boldsymbol{b} = 0 \implies \boldsymbol{a} \perp \boldsymbol{b}$
6. $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2 = a^2$

## Angle between vectors

$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{a_1 b_1 + a_2 b_2}{|\boldsymbol{a}||\boldsymbol{b}|}$$

On CAS: `angle([a b c], [a b c])`

(Action → Vector →Angle)

## Angle between vector and axis

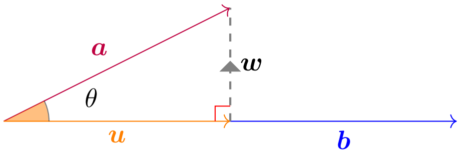
For $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ which makes angles $\alpha, \beta, \gamma$ with positive side of $x, y, z$ axes:

$$\cos\alpha = \frac{a_1}{|\boldsymbol{a}|}, \quad \cos\beta = \frac{a_2}{|\boldsymbol{a}|}, \quad \cos\gamma = \frac{a_3}{|\boldsymbol{a}|}$$

On CAS: `angle([a b c], [1 0 0])`

for angle between $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ and $x$-axis

## Projections & resolutes




### $\parallel \boldsymbol{b}$ (vector projection/resolute)

$$\begin{aligned}\boldsymbol{u} &= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|^2}\boldsymbol{b} \\ &= \left(\boldsymbol{a}\cdot\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right)\left(\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right) \\ &= (\boldsymbol{a}\cdot\hat{\boldsymbol{b}})\hat{\boldsymbol{b}}\end{aligned}$$

### $\perp \boldsymbol{b}$ (perpendicular projection)

$$\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u}$$

### $|\boldsymbol{u}|$ (scalar resolute)

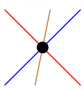
$$\begin{aligned}r_s &= |\boldsymbol{u}| \\ &= \boldsymbol{a}\cdot\hat{\boldsymbol{b}} \\ &= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|}\end{aligned}$$

### Rectangular $(\parallel, \perp)$ components

$$\boldsymbol{a} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}\cdot\boldsymbol{b}}\boldsymbol{b} + \left(\boldsymbol{a} - \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}\cdot\boldsymbol{b}}\boldsymbol{b}\right)$$

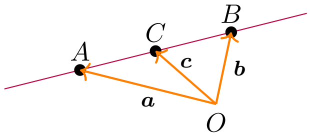
## Vector proofs

**Concurrent:** intersection of $\geq 3$ lines



### Collinear points

$\geq 3$ points lie on the same line




e.g. Prove that

$$\begin{aligned}\overrightarrow{AC} = m\overrightarrow{AB} \iff \boldsymbol{c} &= (1-m)\boldsymbol{a} + m\boldsymbol{b} \\ \implies \boldsymbol{c} &= \overrightarrow{OA} + \overrightarrow{AC} \\ &= \overrightarrow{OA} + m\overrightarrow{AB} \\ &= \boldsymbol{a} + m(\boldsymbol{b} - \boldsymbol{a}) \\ &= \boldsymbol{a} + m\boldsymbol{b} - m\boldsymbol{a} \\ &= (1-m)\boldsymbol{a} + m\boldsymbol{b}\end{aligned}$$

$$\text{Also,} \implies \overrightarrow{OC} = \lambda\vec{OA} + \mu\overrightarrow{OB}$$

$$\text{where } \lambda + \mu = 1$$

$$\text{If } C \text{ lies along } \overrightarrow{AB}, \implies 0 < \mu < 1$$

### Useful vector properties

- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then $\boldsymbol{b} = k\boldsymbol{a}$ for some $k \in \mathbb{R} \setminus \{0\}$
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel with at least one point in common, then they lie on the same straight line
- Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if $\boldsymbol{a}\cdot\boldsymbol{b} = 0$
- $\boldsymbol{a}\cdot\boldsymbol{a} = |\boldsymbol{a}|^2$

## Linear dependence

Vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are linearly dependent if they are non-parallel and:

$$k\boldsymbol{a} + l\boldsymbol{b} + m\boldsymbol{c} = 0$$

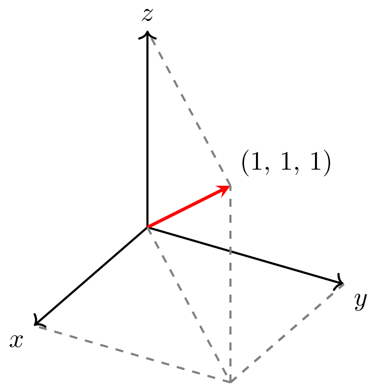
$$\therefore \boldsymbol{c} = m\boldsymbol{a} + n\boldsymbol{b} \quad \text{(simultaneous)}$$

$\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$ are linearly independent if no vector in the set is expressible as a linear combination of other vectors in set, or if they are parallel.

Vector $\boldsymbol{w}$ is a linear combination of vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$

## Three-dimensional vectors

Right-hand rule for axes: $z$ is up or out of page.




## Parametric vectors

Parametric equation of line through point $(x_0, y_0, z_0)$ and parallel to $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ is:

$$\begin{cases} x = x_o + a \cdot t \\ y = y_0 + b \cdot t \\ z = z_0 + c \cdot t \end{cases} \tag{1}$$