Year 12 Specialist

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1   Complex numbers

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

Cartesian form: $a + bi$

Polar form: $r \operatorname{cis} \theta$

Operations

	Cartesian	**Polar**
$z_1 \pm z_2$	$(a \pm c)(b \pm d)i$	convert to $a + bi$
$+k \times z$	$ka \pm kbi$	$kr \operatorname{cis} \theta$
$-k \times z$		$kr \operatorname{cis}(\theta \pm \pi)$
$z_1 \cdot z_2$	$ac - bd + (ad + bc)i$	$r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$
$z_1 \div z_2$	$(z_1 \overline{z_2}) \div |z_2|^2$	$\left(\frac{r_1}{r_2}\right) \operatorname{cis}(\theta_1 - \theta_2)$

Scalar multiplication in polar form

For $k \in \mathbb{R}^+$:

$$k \left(r \operatorname{cis} \theta\right) = kr \operatorname{cis} \theta$$

For $k \in \mathbb{R}^-$:

$$k \left(r \operatorname{cis} \theta\right) = kr \operatorname{cis} \left(\begin{cases} \theta - \pi & |0 < \operatorname{Arg}(z) \le \pi \\ \theta + \pi & |-\pi < \operatorname{Arg}(z) \le 0 \end{cases} \right)$$

Conjugate

`conjg(a+bi)`

$$\overline{z} = a \mp bi$$

$$= r \operatorname{cis}(-\theta)$$

Properties

$$\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$$

$$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$$

$$\overline{kz} = k\overline{z} \, \forall \, k \in \mathbb{R}$$

$$z\overline{z} = (a + bi)(a - bi)$$

$$= a^2 + b^2$$

$$= |z|^2$$

Modulus

$$|z| = |\vec{Oz}| = \sqrt{a^2 + b^2}$$

Properties

$$|z_1 z_2| = |z_1||z_2|$$

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$$

$$|z_1 + z_2| \le |z_1| + |z_2|$$

Multiplicative inverse

$$z^{-1} = \frac{a - bi}{a^2 + b^2}$$

$$= \frac{\overline{z}}{|z|^2} a$$

$$= r \operatorname{cis}(-\theta)$$

Dividing over $\mathbb{C}$

$$\frac{z_1}{z_2} = z_1 z_2^{-1}$$

$$= \frac{z_1 \overline{z_2}}{|z_2|^2}$$

$$= \frac{(a + bi)(c - di)}{c^2 + d^2}$$

then rationalise denominator

Polar form

$$r \operatorname{cis} \theta = r \left(\cos \theta + i \sin \theta\right)$$

- $r = |z| = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}$

- $\theta = \arg(z)$ `arg(a+bi)`

- $\operatorname{Arg}(z) \in (-\pi, \pi)$ **(principal argument)**

- Multiple representations:
  $r \operatorname{cis} \theta = r \operatorname{cis}(\theta + 2n\pi)$ with $n \in \mathbb{Z}$ revolutions

- $\operatorname{cis} \pi = -1,$ $\operatorname{cis} 0 = 1$

On CAS

`compToTrig(a+bi)` $\iff$ `cExpand{r·cisX}`

de Moivres' theorem

$$(r \operatorname{cis} \theta)^n = r^n \operatorname{cis}(n\theta) \text{ where } n \in \mathbb{Z}$$

Complex polynomials

Include $\pm$ for all solutions, incl. imaginary

Sum of squares	$z^2 + a^2 = z^2 - (ai)^2$ $= (z + ai)(z - ai)$
Sum of cubes	$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
Division	$P(z) = D(z)Q(z) + R(z)$
Remainder theorem	Let $\alpha \in \mathbb{C}$. Remainder of $P(z) \div (z - \alpha)$ is $P(\alpha)$
Factor theorem	$z - \alpha$ is a factor of $P(z) \iff$ $P(\alpha) = 0$ for $\alpha \in \mathbb{C}$
Conjugate root theorem	$P(z) = 0$ at $z = a \pm bi$ ($\implies$ both z_1 and $\overline{z_1}$ are solutions)

> **Factor theorem**
>
> If $\beta z + \alpha$ is a factor of $P(z)$,
>   then $P(-\frac{\alpha}{\beta}) = 0$.

nth roots

nth roots of $z = r \operatorname{cis} \theta$ are:

$$z = r^{\frac{1}{n}} \operatorname{cis}\left(\frac{\theta + 2k\pi}{n}\right)$$

- Same modulus for all solutions

- Arguments separated by $\frac{2\pi}{n}$ $\therefore$ there are n roots

- If one square root is $a + bi$, the other is $-a - bi$

- Give one implicit nth root z_1, function is $z = z_1^n$

- Solutions of $z^n = a$ where $a \in \mathbb{C}$ lie on the circle
  $x^2 + y^2 = \left(|a|^{\frac{1}{n}}\right)^2$　　(intervals of $\frac{2\pi}{n}$)

For $0 = az^2 + bz + c$, use quadratic formula:
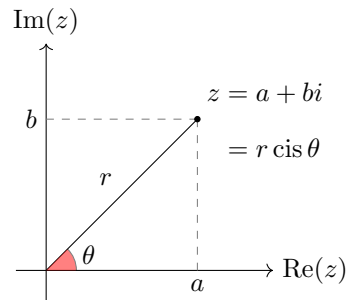
$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Fundamental theorem of algebra

A polynomial of degree n can be factorised into n linear factors in $\mathbb{C}$:

$$\implies P(z) = a_n(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)\ldots(z - \alpha_n)$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \in \mathbb{C}$

Argand planes

- Multiplication by $i \implies$ CCW rotation of $\frac{\pi}{2}$

- Addition: $z_1 + z_2 \equiv \overrightarrow{Oz_1} + \overrightarrow{Oz_2}$

Sketching complex graphs

Linear

- $\operatorname{Re}(z) = c$ or $\operatorname{Im}(z) = c$ (perpendicular bisector)

- $\operatorname{Im}(z) = m \operatorname{Re}(z)$

- $|z + a| = |z + b| \implies 2(a - b)x = b^2 - a^2$
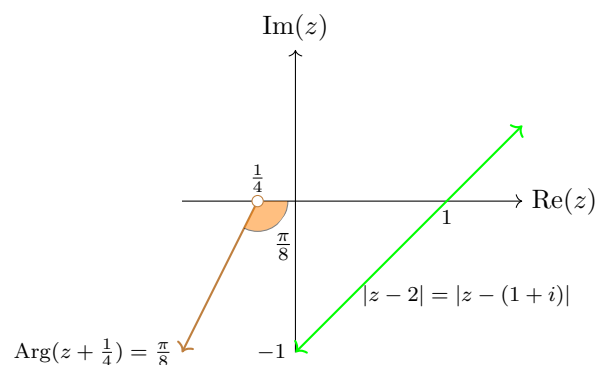 Geometric: equidistant from a, b

Circles

- $|z - z_1|^2 = c^2 |z_2 + 2|^2$
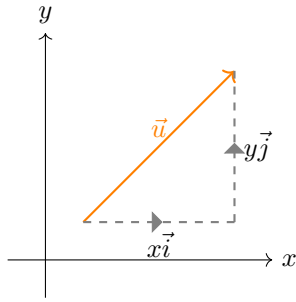
- $|z - (a + bi)| = c \implies (x - a)^2 + (y - b)^2 = c^2$

Loci　　$\operatorname{Arg}(z) < \theta$
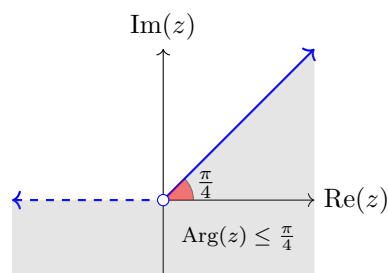
Rays　　$\operatorname{Arg}(z - b) = \theta$

2   Vectors

Column notation

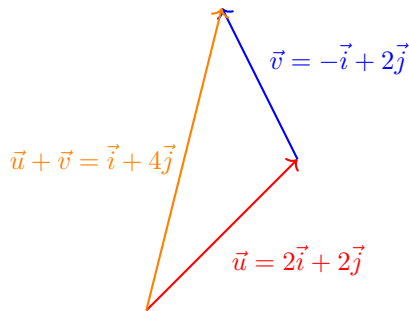$$\begin{bmatrix} x \\ y \end{bmatrix} \iff x\boldsymbol{i} + y\boldsymbol{j}$$

$$\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} \quad \text{between } A(x_1, y_1),\ B(x_2, y_2)$$

Scalar multiplication

$$k \cdot (x\boldsymbol{i} + y\boldsymbol{j}) = kx\boldsymbol{i} + ky\boldsymbol{j}$$

For $k \in \mathbb{R}^-$, direction is reversed

Vector addition

$$(x\boldsymbol{i} + y\boldsymbol{j}) \pm (a\boldsymbol{i} + b\boldsymbol{j}) = (x \pm a)\boldsymbol{i} + (y \pm b)\boldsymbol{j}$$

- Draw each vector head to tail then join lines

- Addition is commutative (parallelogram)

- $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v}) \implies \overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

Magnitude

$$|(x\boldsymbol{i} + y\boldsymbol{j})| = \sqrt{x^2 + y^2}$$

Parallel vectors

$$\boldsymbol{u} || \boldsymbol{v} \iff \boldsymbol{u} = k\boldsymbol{v} \text{ where } k \in \mathbb{R} \setminus \{0\}$$

For parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{cases} |\boldsymbol{a}||\boldsymbol{b}| & \text{if same direction} \\ -|\boldsymbol{a}||\boldsymbol{b}| & \text{if opposite directions} \end{cases}$$

Perpendicular vectors

$$\boldsymbol{a} \perp \boldsymbol{b} \iff \boldsymbol{a} \cdot \boldsymbol{b} = 0 \quad (\text{since } \cos 90 = 0)$$

Unit vector $|\hat{\boldsymbol{a}}| = 1$

$$\hat{a} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$$

$$= \boldsymbol{a} \cdot |\boldsymbol{a}|$$

Scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$
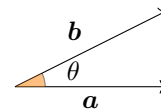
$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$

$$= |\boldsymbol{a}||\boldsymbol{b}| \cos \theta$$

$$(0 \leq \theta \leq \pi) \text{ - from cosine rule}$$

On CAS: `dotP([a b c], [d e f])`

Properties

1. $k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (k\boldsymbol{b})$

2. $\boldsymbol{a} \cdot \boldsymbol{0} = 0$

3. $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$

4. $\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1$

5. $\boldsymbol{a} \cdot \boldsymbol{b} = 0 \implies \boldsymbol{a} \perp \boldsymbol{b}$

6. $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2 = a^2$

Angle between vectors

$$\cos \theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{a_1 b_1 + a_2 b_2}{|\boldsymbol{a}||\boldsymbol{b}|}$$

On CAS: `angle([a b c], [a b c])`

(Action → Vector →Angle)

Angle between vector and axis

For $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ which makes angles α, β, γ with positive side of x, y, z axes:

$$\cos\alpha = \frac{a_1}{|\boldsymbol{a}|}, \quad \cos\beta = \frac{a_2}{|\boldsymbol{a}|}, \quad \cos\gamma = \frac{a_3}{|\boldsymbol{a}|}$$

On CAS: `angle([a b c], [1 0 0])`

for angle between $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ and x-axis

Projections & resolutes
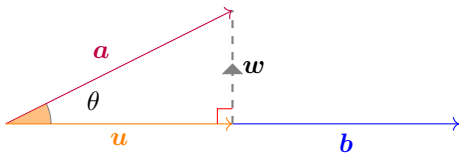
$\parallel \boldsymbol{b}$ (vector projection/resolute)

$$\boldsymbol{u} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|^2}\boldsymbol{b}$$
$$= \left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}\right)\left(\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right)$$
$$= (\boldsymbol{a} \cdot \hat{\boldsymbol{b}})\hat{\boldsymbol{b}}$$

$\perp \boldsymbol{b}$ (perpendicular projection)

$$\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u}$$

$|\boldsymbol{u}|$ (scalar projection/resolute)

$$s = |\boldsymbol{u}|$$
$$= \boldsymbol{a} \cdot \hat{\boldsymbol{b}}$$
$$= \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$$
$$= |\boldsymbol{a}|\cos\theta$$

Rectangular ($\parallel, \perp$) components

$$\boldsymbol{a} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b} + \left(\boldsymbol{a} - \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}\right)$$

Vector proofs

Concurrent: intersection of ≥ 3 lines

Collinear points

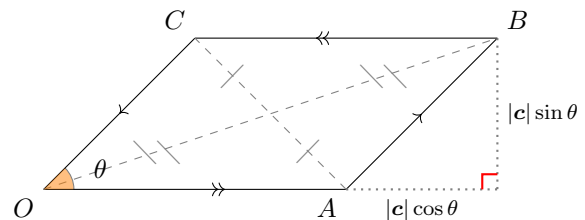≥ 3 points lie on the same line
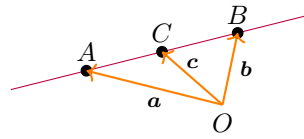
e.g. Prove that

$$\overrightarrow{AC} = m\overrightarrow{AB} \iff \boldsymbol{c} = (1-m)\boldsymbol{a} + m\boldsymbol{b}$$
$$\implies \boldsymbol{c} = \overrightarrow{OA} + \overrightarrow{AC}$$
$$= \overrightarrow{OA} + m\overrightarrow{AB}$$
$$= \boldsymbol{a} + m(\boldsymbol{b} - \boldsymbol{a})$$
$$= \boldsymbol{a} + m\boldsymbol{b} - m\boldsymbol{a}$$
$$= (1-m)\boldsymbol{a} + m\boldsymbol{b}$$

Also, $\implies \overrightarrow{OC} = \lambda\overrightarrow{OA} + \mu\overrightarrow{OB}$

where $\lambda + \mu = 1$

If C lies along $\overrightarrow{AB}, \implies 0 < \mu < 1$

Parallelograms

- Diagonals $\overrightarrow{OB}, \overrightarrow{AC}$ bisect each other

- If diagonals are equal length, it is a rectangle

- $|\overrightarrow{OB}|^2 + |\overrightarrow{CA}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{AB}|^2 + |\overrightarrow{CB}|^2 + |\overrightarrow{OC}|^2$

- Area $= \boldsymbol{c} \cdot \boldsymbol{a}$

Useful vector properties

- $\boldsymbol{a} \parallel \boldsymbol{b} \implies \boldsymbol{b} = k\boldsymbol{a}$ for some $k \in \mathbb{R} \setminus \{0\}$

- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel with at least one point in common, then they lie on the same straight line

- $\boldsymbol{a} \perp \boldsymbol{b} \iff \boldsymbol{a} \cdot \boldsymbol{b} = 0$

- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$

Linear dependence

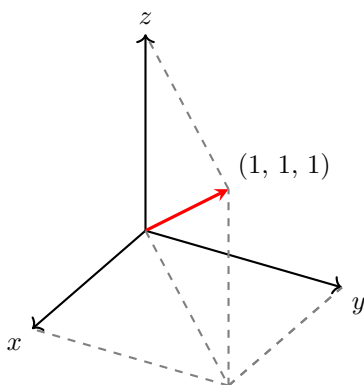a, b, c are linearly dependent if they are $\nparallel$ and:

$$0 = ka + lb + mc$$

$$\therefore c = ma + nb \quad \text{(simultaneous)}$$

a, b, and c are linearly independent if no vector in the set is expressible as a linear combination of other vectors in set, or if they are parallel.

Three-dimensional vectors

Right-hand rule for axes: z is up or out of page.

Parametric vectors

Parametric equation of line through point (x_0, y_0, z_0) and parallel to $ai + bj + ck$ is:

$$\begin{cases} x = x_o + a \cdot t \\ y = y_0 + b \cdot t \\ z = z_0 + c \cdot t \end{cases}$$

3   Circular functions

$\sin(bx)$ or $\cos(bx)$: period $= \frac{2\pi}{b}$

$\tan(nx)$: period $= \frac{\pi}{n}$

asymptotes at $x = \frac{(2k+1)\pi}{2n} \mid k \in \mathbb{Z}$

Reciprocal functions

Cosecant

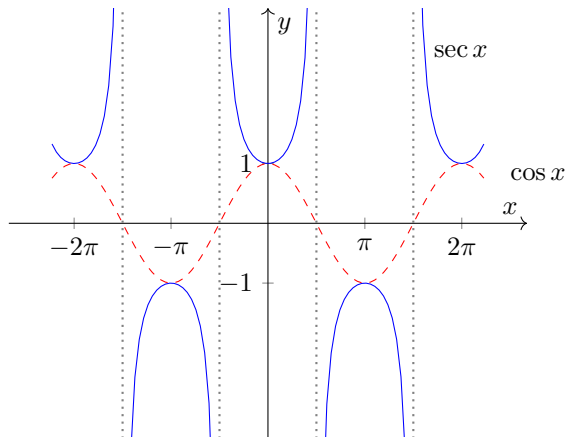$$\operatorname{cosec} \theta = \frac{1}{\sin \theta} \mid \sin \theta \neq 0$$

- **Domain** $= \mathbb{R} \setminus n\pi : n \in \mathbb{Z}$

- **Range** $= \mathbb{R} \setminus (-1, 1)$

- **Turning points** at $\theta = \frac{(2n+1)\pi}{2} \mid n \in \mathbb{Z}$

- **Asymptotes** at $\theta = n\pi \mid n \in \mathbb{Z}$

Secant

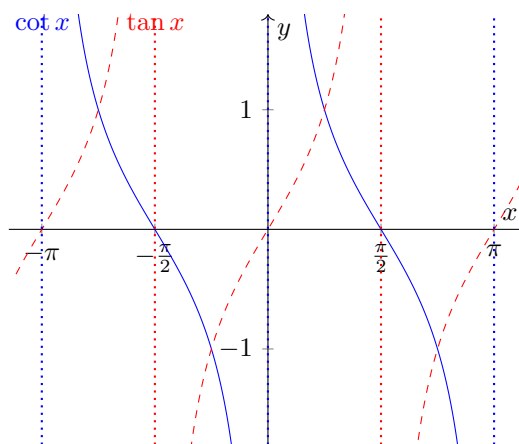$$\sec \theta = \frac{1}{\cos \theta} \mid \cos \theta \neq 0$$

- **Domain** $= \mathbb{R} \setminus \frac{(2n+1)\pi}{2} : n \in \mathbb{Z}\}$

- **Range** $= \mathbb{R} \setminus (-1, 1)$

- **Turning points** at $\theta = n\pi \mid n \in \mathbb{Z}$

- **Asymptotes** at $\theta = \frac{(2n+1)\pi}{2} \mid n \in \mathbb{Z}$

Cotangent

$$\cot \theta = \frac{\cos \theta}{\sin \theta} \mid \sin \theta \neq 0$$

- **Domain** $= \mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$

- **Range** $= \mathbb{R}$

- **Asymptotes** at $\theta = n\pi \mid n \in \mathbb{Z}$

Symmetry properties

$$\sec(\pi \pm x) = -\sec x$$

$$\sec(-x) = \sec x$$

$$\operatorname{cosec}(\pi \pm x) = \mp \operatorname{cosec} x$$

$$\operatorname{cosec}(-x) = -\operatorname{cosec} x$$

$$\cot(\pi \pm x) = \pm \cot x$$

$$\cot(-x) = -\cot x$$

Complementary properties

$$\sec\left(\frac{\pi}{2} - x\right) = \operatorname{cosec} x$$

$$\operatorname{cosec}\left(\frac{\pi}{2} - x\right) = \sec x$$

$$\cot\left(\frac{\pi}{2} - x\right) = \tan x$$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x$$

Pythagorean identities

$$1 + \cot^2 x = \operatorname{cosec}^2 x, \quad \text{where } \sin x \neq 0$$

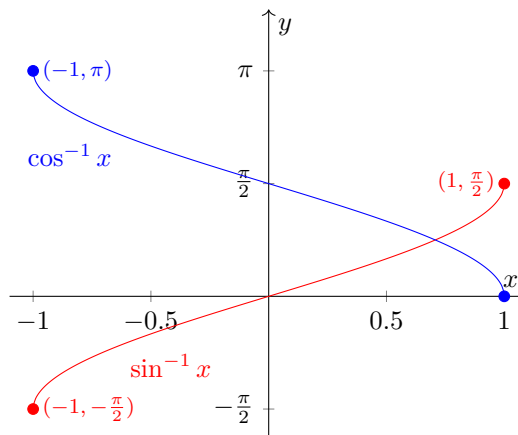$$1 + \tan^2 x = \sec^2 x, \quad \text{where } \cos x \neq 0$$

Compound angle formulas

$$\cos(x \pm y) = \cos x + \cos y \mp \sin x \sin y$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

Double angle formulas

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= 1 - 2\sin^2 x$$

$$= 2\cos^2 x - 1$$

$$\sin 2x = 2\sin x \cos x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Inverse circular functions
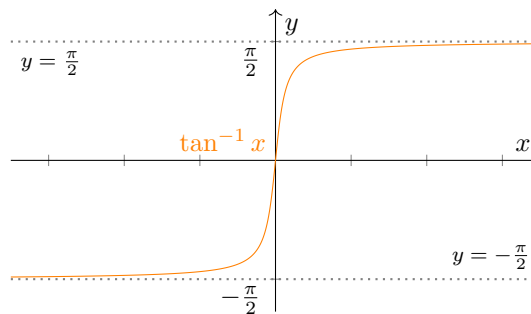

Inverse functions: $f(f^{-1}(x)) = x$ (restrict domain)

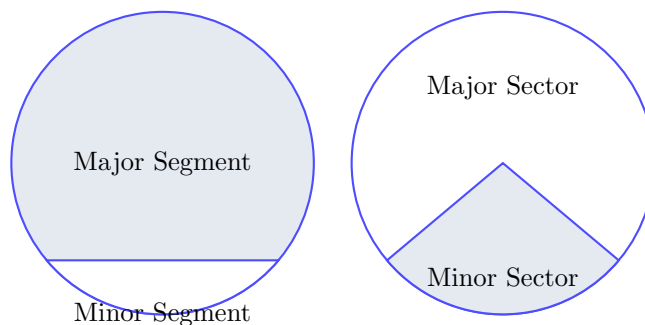$$\sin^{-1} : [-1, 1] \to \mathbb{R}, \quad \sin^{-1} x = y$$

$$\text{where } \sin y = x, \ y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

$$\cos^{-1} : [-1, 1] \to \mathbb{R}, \quad \cos^{-1} x = y$$

$$\text{where } \cos y = x, \ y \in [0, \pi]$$

$$\tan^{-1} : \mathbb{R} \to \mathbb{R}, \quad \tan^{-1} x = y$$

$$\text{where } \tan y = x, \ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Mensuration

$$\textbf{Sectors:}\ A = \pi r^2 \frac{\theta}{2\pi}$$
$$= \frac{r^2 \theta}{2}$$

$$\textbf{Segments:}\ A = \frac{r^2}{2}\,(\theta = \sin\theta)$$

$$\textbf{Chords:}\ \operatorname{crd}\theta = \sqrt{(1-\cos\theta)^2 + \sin^2\theta}$$
$$= \sqrt{2 - 2\cos\theta}$$
$$= 2\sin\left(\frac{\theta}{2}\right)$$

4   Differential calculus

$$f'(x) = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

Limits

$$\lim_{x \to a} f(x)$$

L^-, L^+ limit from below/above

$\lim_{x \to a} f(x)$ limit of a point

For solving $x \to \infty$, put all x terms in denominators
e.g.

$$\lim_{x \to \infty} \frac{2x + 3}{x - 2} = \frac{2 + \frac{3}{x}}{1 - \frac{2}{x}} = \frac{2}{1} = 2$$

Limit theorems

1. For constant function $f(x) = k$, $\lim_{x \to a} f(x) = k$

2. $\lim_{x \to a}(f(x) \pm g(x)) = F \pm G$

3. $\lim_{x \to a}(f(x) \times g(x)) = F \times G$

4. $\therefore \lim_{x \to a} c \times f(x) = cF$ where $c = $ constant

5. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G}, G \neq 0$

6. $f(x)$ is continuous $\iff L^- = L^+ = f(x)\,\forall x$

Gradients

Secant (chord) - line joining two points on curve

Tangent - line that intersects curve at one point

Points of Inflection

Stationary point - i.e. $f'(x) = 0$

Point of inflection - max |gradient| (i.e. $f'' = 0$)

Strictly increasing/decreasing

For x_2 and x_1 where $x_2 > x_1$:

strictly increasing

 where $f(x_2) > f(x_1)$ or $f'(x) > 0$

strictly decreasing

where $f(x_2) < f(x_1)$ or $f'(x) < 0$
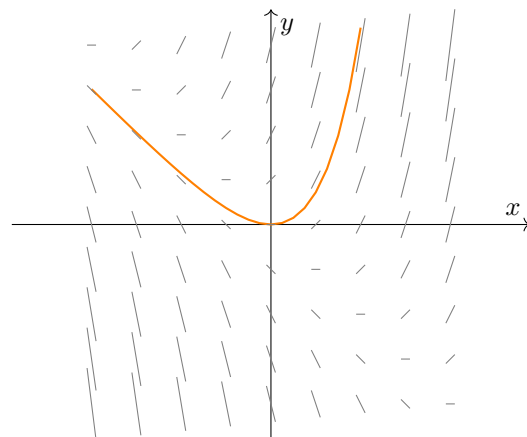
> Endpoints are included, even where $\frac{dy}{dx} = 0$

Second derivative
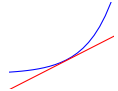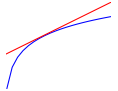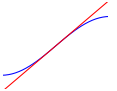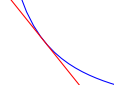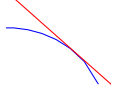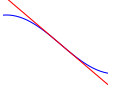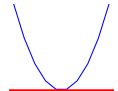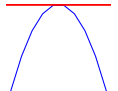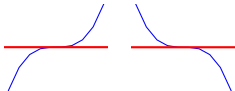
$$f(x) \longrightarrow f'(x) \longrightarrow f''(x)$$
$$\implies y \longrightarrow \frac{dy}{dx} \longrightarrow \frac{d^2 y}{dx^2}$$

Order of polynomial nth derivative decrements each time the derivative is taken

Slope fields

- $f'(a) = 0$, $f''(a) > 0$

  local min at $(a, f(a))$ (concave up)

- $f'(a) = 0$, $f''(a) < 0$

  local max at $(a, f(a))$ (concave down)

- $f''(a) = 0$

  point of inflection at $(a, f(a))$

- $f''(a) = 0$, $f'(a) = 0$

  stationary point of inflection at $(a, f(a)$

	$\dfrac{d^2y}{dx^2} > 0$	$\dfrac{d^2y}{dx^2} < 0$	$\dfrac{d^2y}{dx^2} = 0$ (inflection)
$\dfrac{dy}{dx} > 0$	Rising (concave up)	Rising (concave down)	Rising inflection point
$\dfrac{dy}{dx} < 0$	Falling (concave up)	Falling (concave down)	Falling inflection point
$\dfrac{dy}{dx} = 0$	Local minimum	Local maximum	Stationary inflection point

Implicit Differentiation

Used for differentiating circles etc.

If p and q are expressions in x and y such that $p = q$, for all x and y, then:

$$\frac{dp}{dx} = \frac{dq}{dx} \quad \text{and} \quad \frac{dp}{dy} = \frac{dq}{dy}$$

On CAS

Action → Calculation

```
impDiff(y^2+ax=5, x, y)
```

Function of the dependent variable

If $\frac{dy}{dx} = g(y)$, then $\frac{dx}{dy} = 1 \div \frac{dy}{dx} = \frac{1}{g(y)}$. Integrate both sides to solve equation. Only add c on one side. Express e^c as A.

Reciprocal derivatives

$$\frac{1}{\frac{dy}{dx}} = \frac{dx}{dy}$$

Differentiating $x = f(y)$

Find $\dfrac{dx}{dy}$, then $\dfrac{dy}{dx} = \dfrac{1}{\left(\dfrac{dx}{dy}\right)}$

Parametric equations

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} \quad \text{provided } \frac{dx}{dt} \neq 0$$

$$\frac{d^2y}{dx^2} = \frac{\left(\dfrac{dy'}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} \quad \text{where } y' = \frac{dy}{dx}$$

Integration

$$\int f(x) \cdot dx = F(x) + c \quad \text{where } F'(x) = f(x)$$

Properties

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

$$\int_a^a f(x)\, dx = 0$$

$$\int_a^b k \cdot f(x)\, dx = k \int_a^b f(x)\, dx$$

$$\int_a^b f(x) \pm g(x)\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$$

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

Integration by substitution

$$\int f(u)\frac{du}{dx} \cdot dx = \int f(u) \cdot du$$

> $f(u)$ **must be 1:1** $\implies$ **one x for each y**

e.g. for $y = \int (2x + 1)\sqrt{x + 4} \cdot dx$

let $u = x + 4$

$\implies \dfrac{du}{dx} = 1$

$\implies x = u - 4$

then $y = \int (2(u - 4) + 1)u^{\frac{1}{2}} \cdot du$

(solve as normal integral)

Definite integrals by substitution

For $\int_a^b f(x)\frac{du}{dx} \cdot dx$, evaluate new a and b for $f(u) \cdot du$.

Trigonometric integration

$$\sin^m x \cos^n x \cdot dx$$

m **is odd:** $m = 2k + 1$ where $k \in \mathbb{Z}$

$\implies \sin^{2k+1} x = (\sin^2 z)^k \sin x = (1 - \cos^2 x)^k \sin x$

Substitute $u = \cos x$

n **is odd:** $n = 2k + 1$ where $k \in \mathbb{Z}$

$\implies \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x$

Substitute $u = \sin x$

m **and n are even:** use identities...

- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

- $\sin 2x = 2\sin x \cos x$

Separation of variables

If $\frac{dy}{dx} = f(x)g(y)$, then:

$$\int f(x)\, dx = \int \frac{1}{g(y)}\, dy$$

Partial fractions

To factorise $f(x) = \frac{\delta}{\alpha \cdot \beta}$:

$$\frac{\delta}{\alpha \cdot \beta \cdot \gamma} = \frac{A}{\alpha} + \frac{B}{\beta} + \frac{C}{\gamma} \tag{1}$$

Multiply by $(\alpha \cdot \beta \cdot \gamma)$:

$$\delta = \beta\gamma A + \alpha\gamma B + \alpha\beta C \tag{2}$$

Substitute $x = \{\alpha, \beta, \gamma\}$ into (2) to find denominators

Repeated linear factors

$$\frac{p(x)}{(x - a)^n} = \frac{A_1}{(x - a)} + \frac{A_2}{(x - a)^2} + \cdots + \frac{A_n}{(x - a)^n}$$

Irreducible quadratic factors

e.g. $\dfrac{3x - 4}{(2x - 3)(x^2 + 5)} = \dfrac{A}{2x - 3} + \dfrac{Bx + C}{x^2 + 5}$

> **On CAS**
>
> Action → Transformation:
>
>   `expand(..., x)`
>
> To reverse, use `combine(...)`

Integrating $\dfrac{dy}{dx} = g(y)$

$$\text{if } \frac{dy}{dx} = g(y), \text{ then } x = \int \frac{1}{g(y)}\, dy$$

Graphing integrals on CAS

> **On CAS**
>
> **In main:** Interactive → Calculation → $\int$
>
> For restrictions, `Define f(x)=...` then
>
> `f(x)|x>...`

Solids of revolution

Approximate as sum of infinitesimally-thick cylinders

Rotation about x-axis

$$V = \pi \int_{x=a}^{x=b} f(x)^2\, dx$$

Rotation about y-axis

$$V = \pi \int_{y=a}^{y=b} x^2\, dy$$

$$= \pi \int_{y=a}^{y=b} (f(y))^2\, dy$$

Regions not bound by $y = 0$

$$V = \pi \int_a^b f(x)^2 - g(x)^2 \, dx$$

$$\text{where } f(x) > g(x)$$

Length of a curve

For length of $f(x)$ from $x = a \to x = b$:

Cartesian $L = \int_a^b \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} \, dx$

Parametric $L = \int_a^b \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2} \, dt$

On CAS

a) Evaluate formula

b) Interactive $\to$ Calculation $\to$ Line $\to$ `arcLen`

Applications of antidifferentiation

- x-intercepts of $y = f(x)$ identify x-coordinates of stationary points on $y = F(x)$

- nature of stationary points is determined by sign of $y = f(x)$ on either side of its x-intercepts

- if $f(x)$ is a polynomial of degree n, then $F(x)$ has degree $n + 1$

To find stationary points of a function, substitute x value of given point into derivative. Solve for $\frac{dy}{dx} = 0$. Integrate to find original function.

Rates

Gradient at a point on parametric curve

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} \mid \frac{dx}{dt} \neq 0 \text{ (chain rule)}$$

$$\frac{d^2}{dx^2} = \frac{d(y')}{dx} = \frac{dy'}{dt} \div \frac{dx}{dt} \mid y' = \frac{dy}{dx}$$

Rational functions

$$f(x) = \frac{P(x)}{Q(x)} \quad \text{where } P, Q \text{ are polynomial functions}$$

Fundamental theorem of calculus

If f is continuous on $[a, b]$, then

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

$$\text{where } F = \int f \, dx$$

Differential equations

Order - highest power inside derivative

Degree - highest power of highest derivative

e.g. $\left(\dfrac{dy^2}{d^2} x\right)^3$ order 2, degree 3

> To verify solutions, find $\frac{dy}{dx}$ from y and substitute into original

Mixing problems

$$\left(\frac{dm}{dt}\right)_\Sigma = \left(\frac{dm}{dt}\right)_{\text{in}} - \left(\frac{dm}{dt}_{\text{out}}\right)$$

Euler's method

$$\frac{f(x + h) - f(x)}{h} \approx f'(x) \quad \text{for small } h$$

$$\implies f(x + h) \approx f(x) + h f'(x)$$

> If $\dfrac{dy}{dx} = g(x)$ with $x_0 = a$ and $y_0 = b$, then:
>
> $$x_{n+1} = x_n + h$$
> $$y_{n+1} = y_n + h g(x_n)$$

Derivatives

$f(x)$	$f'(x)$	
$\sin x$	$\cos x$	
$\sin ax$	$a \cos ax$	
$\cos x$	$-\sin x$	
$\cos ax$	$-a \sin ax$	
$\tan f(x)$	$f^2(x) \sec^2 f(x)$	
e^x	e^x	
e^{ax}	ae^{ax}	
ax^{nx}	$an \cdot e^{nx}$	
$\log_e x$	$\dfrac{1}{x}$	
$\log_e ax$	$\dfrac{1}{x}$	
$\log_e f(x)$	$\dfrac{f'(x)}{f(x)}$	
$\sin(f(x))$	$f'(x) \cdot \cos(f(x))$	
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$	
$\cos^{-1} x$	$\dfrac{-1}{\sqrt{1-x^2}}$	
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$	
$\dfrac{d}{dy} f(y)$	$\dfrac{1}{\frac{dx}{dy}}$	(reciprocal)
uv	$u\frac{dv}{dx} + v\frac{du}{dx}$	(product rule)
$\dfrac{u}{v}$	$\dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	(quotient rule)
$f(g(x))$	$f'(g(x)) \cdot g'(x)$	

Antiderivatives

$f(x)$	$\int f(x) \cdot dx$	
k (constant)	$kx + c$	
x^n	$\dfrac{1}{n+1} x^{n+1}$	
ax^{-n}	$a \cdot \log_e |x| + c$	
$\dfrac{1}{ax+b}$	$\dfrac{1}{a} \log_e(ax+b) + c$	
$(ax+b)^n$	$\dfrac{1}{a(n+1)}(ax+b)^{n-1} + c \mid n \neq 1$	
$(ax+b)^{-1}$	$\dfrac{1}{a} \log_e |ax+b| + c$	
e^{kx}	$\dfrac{1}{k} e^{kx} + c$	
e^k	$e^k x + c$	
$\sin kx$	$\dfrac{-1}{k} \cos(kx) + c$	
$\cos kx$	$\dfrac{1}{k} \sin(kx) + c$	
$\sec^2 kx$	$\dfrac{1}{k} \tan(kx) + c$	
$\dfrac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1} \dfrac{x}{a} + c \mid a > 0$	
$\dfrac{-1}{\sqrt{a^2 - x^2}}$	$\cos^{-1} \dfrac{x}{a} + c \mid a > 0$	
$\dfrac{a}{a^2 - x^2}$	$\tan^{-1} \dfrac{x}{a} + c$	
$\dfrac{f'(x)}{f(x)}$	$\log_e f(x) + c$	
$\int f(u) \cdot \frac{du}{dx} \cdot dx$	$\int f(u) \cdot du$	(substitution)
$f(x) \cdot g(x)$	$\int [f'(x) \cdot g(x)]dx + \int [g'(x)f(x)]dx$	

Note $\sin^{-1}\left(\dfrac{x}{a}\right) + \cos^{-1}\left(\dfrac{x}{a}\right)$ is constant $\forall\, x \in (-a, a)$

Index identities

$$b^{m+n} = b^m \cdot b^n$$

$$(b^m)^n = b^{m \cdot n}$$

$$(b \cdot c)^n = b^n \cdot c^n$$

$$a^m \div a^n = a^{m-n}$$

Logarithmic identities

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$$

$$\log_b x^n = n \log_b x$$

$$\log_b y^{x^n} = x^n \log_b y$$

5   Kinematics & Mechanics

Constant acceleration

- **Position** - relative to origin

- **Displacement** - relative to starting point

Velocity-time graphs

Displacement: *signed* area

Distance travelled: *total* area

$$\text{acceleration} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

	no
$v = u + at$	x
$v^2 = u^2 + 2as$	t
$s = \frac{1}{2}(v+u)t$	a
$s = ut + \frac{1}{2}at^2$	v
$s = vt - \frac{1}{2}at^2$	u

$$v_{\text{avg}} = \frac{\Delta\text{position}}{\Delta t}$$

$$\text{speed} = |\text{velocity}|$$
$$= \sqrt{v_x^2 + v_y^2 + v_z^2}$$

Distance travelled between $t = a \to t = b$:

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \text{(2D)}$$

$$= \int_{t=a}^{t=b} \frac{dx}{dt}\, dt \qquad \text{(linear)}$$

Shortest distance between $r(t_0)$ and $r(t_1)$:

$$= |r(t_1) - r(t_2)|$$

Vector functions

$$r(t) = xi + yj + zk$$

- If $r(t) \equiv$ position with time, then the graph of endpoints of $r(t) \equiv$ Cartesian path

- Domain of $r(t)$ is the range of $x(t)$

- Range of $r(t)$ is the range of $y(t)$

Vector calculus

Derivative

Let $r(t) = x(t)i + y(t)(j)$. If both $x(t)$ and $y(t)$ are differentiable, then:

$$r(t) = x(t)i + y(t)j$$

6   Dynamics

Resolution of forces

Resultant force is sum of force vectors

In angle-magnitude form

Cosine rule: $\quad c^2 \;=\; a^2 \;+\; b^2 \;-\; 2ab\cos\theta$

Sine rule: $\quad \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

In i—j form

Vector of a N at θ to x axis is equal to $a\cos\theta i + a\sin\theta j$. Convert all force vectors then add.

To find angle of an $ai + bj$ vector, use $\theta = \tan^{-1}\frac{b}{a}$

Resolving in a given direction

The resolved part of a force P at angle θ is has magnitude $P\cos\theta$

To convert force $||\vec{OA}$ to angle-magnitude form, find component $\perp \vec{OA}$ then:

$$|r| = \sqrt{\left(||\vec{OA}\right)^2 + \left(\perp \vec{OA}\right)^2}$$

$$\theta = \tan^{-1}\frac{\perp \vec{OA}}{||\vec{OA}}$$

Newton's laws

1. Velocity is constant without ΣF

2. $\frac{d}{dt}\rho \propto \Sigma F \implies F = ma$

3. Equal and opposite forces

Weight

A mass of m kg has force of mg acting on it

Momentum ρ

$$\rho = mv \qquad \text{(units kg m/s or Ns)}$$

Reaction force R

- With no vertical velocity, $R = mg$

- With vertical acceleration, $|R| = m|a| - mg$

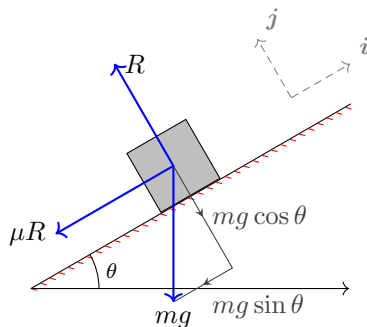- With force F at angle θ, then $R = mg - F\sin\theta$

Friction
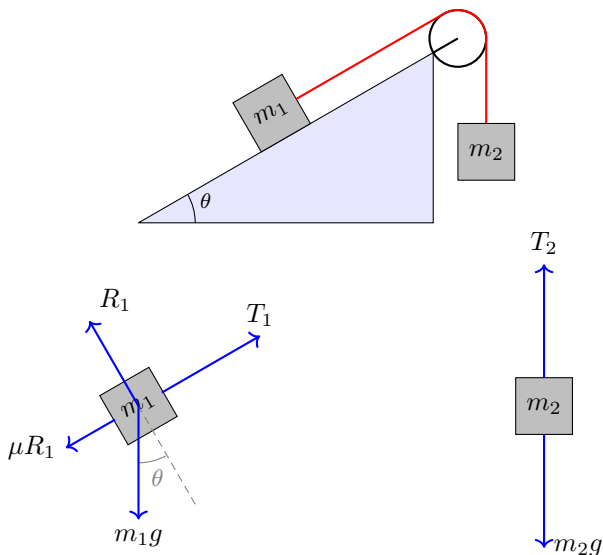
$$F_R = \mu R \qquad \text{(friction coefficient)}$$

Inclined planes

$$\boldsymbol{F} = |\boldsymbol{F}|\cos\theta\boldsymbol{i} + |\boldsymbol{F}|\sin\theta\boldsymbol{j}$$

- Normal force R is at right angles to plane

- Let direction up the plane be $\boldsymbol{i}$ and perpendicular to plane $\boldsymbol{j}$

Connected particles

- **Suspended pulley:** $T_1 = T_2$

  $$|a| = g\frac{m_1 - m_2}{m_1 + m_2} \text{ where } m_1 \text{ accelerates down}$$

  $$\left.\begin{cases} m_1g - T = m_1a \\[2mm] T - m_2g = m_2a \end{cases}\right\} \implies m_1g - m_2g = m_1a + m_2a$$
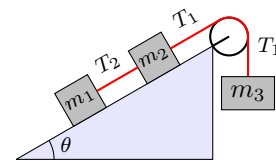
- **String pulling mass on inclined pane:** Resolve parallel to plane

  $$T - mg\sin\theta = ma$$

- **Linear connection:** find acceleration of system first

- **Pulley on right angle:** $a = \frac{m_2g}{m_1 + m_2}$ where m_2 is suspended (frictionless on both surfaces)

- **Pulley on edge of incline:** find downwards force W_2 and components of mass on plane

In this example, note $T_1 \neq T_2$:

Equilibrium

$$\frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c} \qquad \text{(Lami's theorem)}$$

$$c^2 = a^2 + b^2 - 2ab\cos\theta \qquad \text{(cosine rule)}$$

Three methods:

1. Lami's theorem (sine rule)

2. Triangle of forces (cosine rule)

3. Resolution of forces ($\Sigma F = 0$ - simultaneous)

On CAS

To verify: Geometry tab, then select points with normal cursor. Click right arrow at end of toolbar and input point, then lock known constants.

Variable forces (DEs)

$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

7   Statistics

Continuous random variables

A continuous random variable X has a pdf f such that:

1. $f(x) \geq 0 \forall x$

2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$E(X) = \int_{\mathbf{X}} (x \cdot f(x))\, dx$$

$$\text{Var}(X) = E\left[(X - \mu)^2\right]$$

$$\Pr(X \leq c) = \int_{-\infty}^{c} f(x)\, dx$$

Two random variables X, Y

If X and Y are independent:

$$\text{E}(aX + bY) = a\,\text{E}(X) + b\,\text{E}(Y)$$

$$\text{Var}(aX \pm bY \pm c) = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y)$$

Linear functions $X \to aX + b$

$$\Pr(Y \leq y) = \Pr(aX + b \leq y)$$

$$= \Pr\left(X \leq \frac{y - b}{a}\right)$$

$$= \int_{-\infty}^{\frac{y-b}{a}} f(x)\, dx$$

Mean:	$\text{E}(aX + b) = a\,\text{E}(X) + b$
Variance:	$\text{Var}(aX + b) = a^2\,\text{Var}(X)$

Expectation theorems

For some non-linear function g, the expected value $E(g(X))$ is not equal to $g(E(X))$.

$$E(X^2) = \text{Var}(X) - [E(X)]^2$$

$$E(X^n) = \Sigma x^n \cdot p(x) \qquad \text{(non-linear)}$$

$$\neq [E(X)]^n$$

$$E(aX \pm b) = aE(X) \pm b \qquad \text{(linear)}$$

$$E(b) = b \qquad (\forall b \in \mathbb{R})$$

$$E(X + Y) = E(X) + E(Y) \qquad \text{(two variables)}$$

Sample mean

Approximation of the **population mean** determined experimentally.

$$\overline{x} = \frac{\Sigma x}{n}$$

where

n is the size of the sample (number of sample points)

x is the value of a sample point

> **On CAS**
>
> 1. Spreadsheet
>
> 2. In cell A1:
>
>   `mean(randNorm(sd, mean, sample size))`
>
> 3. Edit → Fill → Fill Range
>
> 4. Input range as A1:An where n is the number of samples
>
> 5. Graph → Histogram

Sample size of n

$$\overline{X} = \sum_{i=1}^{n} \frac{x_i}{n} = \frac{\sum x}{n}$$

Sample mean is distributed with mean μ and sd $\frac{\sigma}{\sqrt{n}}$ (approaches these values for increasing sample size n).

For a new distribution with mean of n trials, $\text{E}(X') = \text{E}(X)$, $\quad \text{sd}(X') = \dfrac{\text{sd}(X)}{\sqrt{n}}$

> **On CAS**
>
> 　　Spreadsheet　　→　　Catalog　　→
> `randNorm(sd, mean, n)`
> where **n** is the number of samples. Show histogram
> with Histogram key in top left.
> To calculate parameters of a dataset:
> 　Calc → One-variable

$\overline{x}$ is the sample mean

σ is the population sd

n is the sample size from which $\overline{x}$ was calculated

> **On CAS**
>
> Menu → Stats → Calc → Interval
> Set *Type = One-Sample Z Int*
> 　and select *Variable*

Normal distributions

$$Z = \frac{X - \mu}{\sigma}$$

Normal distributions must have area (total prob.) of 1
$\implies \int_{-\infty}^{\infty} f(x)\,dx = 1$

mean = mode = median

> **Always express z as +ve. Express confidence interval as ordered pair.**

Central limit theorem

> If X is randomly distributed with mean μ and sd σ, then with an adequate sample size n the distribution of the sample mean $\overline{X}$ is approximately normal with mean $E(\overline{X})$ and $\text{sd}(\overline{X}) = \frac{\sigma}{\sqrt{n}}$.

Confidence intervals

- **Point estimate:** single-valued estimate of the population mean from the value of the sample mean $\overline{x}$

- **Interval estimate:** confidence interval for population mean μ

- $C\%$ confidence interval $\implies$ $C\%$ of samples will contain population mean μ

95% confidence interval

For 95% c.i. of population mean μ:

$$x \in \left(\overline{x} \pm 1.96 \frac{\sigma}{\sqrt{n}} \right)$$

where:

Margin of error

For 95% confidence interval of μ:

$$M = 1.96 \times \frac{\sigma}{\sqrt{n}}$$
$$\implies n = \left(\frac{1.96\sigma}{M} \right)^2$$

Always round n up to a whole number of samples.

General case

For $C\%$ c.i. of population mean μ:

$$x \in \left(\overline{x} \pm k \frac{\sigma}{\sqrt{n}} \right)$$

where k is such that $\Pr(-k < Z < k) = \frac{C}{100}$

Confidence interval for multiple trials

For a set of n confidence intervals (samples), there is 0.95^n chance that all n intervals contain the population mean μ.

8　Hypothesis testing

> **Note hypotheses are always expressed in terms of population parameters**

Null hypothesis H$_0$

Sample drawn from population has same mean as control population, and any difference can be explained by sample variations.

Alternative hypothesis $\mathbf{H}_1$

Amount of variation from control is significant, despite standard sample variations.

p-value

Probability of observing a value of the sample statistic as significant as the one observed, assuming null hypothesis is true.

For one-tail tests:

$$p\text{-value} = \Pr\left(\, \overline{X} \lesseqgtr \mu(\mathbf{H}_1) \mid \mu = \mu(\mathbf{H}_0) \,\right)$$
$$= \Pr\left(Z \lesseqgtr \frac{(\mu(\mathbf{H}_1) - \mu(\mathbf{H}_0)) \cdot \sqrt{n}}{\text{sd}(X)} \right)$$

then use `normCdf` with std. norm.

p	Conclusion
> 0.05	insufficient evidence against $\mathbf{H}_0$
< 0.05 (5%)	good evidence against $\mathbf{H}_0$
< 0.01 (1%)	strong evidence against $\mathbf{H}_0$
< 0.001 (0.1%)	very strong evidence against $\mathbf{H}_0$

Finding n for a given p-value

Find c such that $\Pr(Z \lesseqgtr c)$ such that $c = \alpha$ (use `invNormCdf` on CAS).

Significance level α

The condition for rejecting the null hypothesis.

If $p < \alpha$, null hypothesis is **rejected**

If $p > \alpha$, null hypothesis is **accepted**

z-test

Hypothesis test for a mean of a sample drawn from a normally distributed population with a known standard deviation.

> **On CAS**
>
> Menu $\rightarrow$ Statistics $\rightarrow$ Calc $\rightarrow$ Test.
>
> Select *One-Sample Z-Test* and *Variable*, then input:
>
> | | |
> |---|---|
> | μ cond: | same operator as $\mathbf{H}_1$ |
> | μ_0: | expected sample mean (null hypothesis) |
> | σ: | standard deviation (null hypothesis) |
> | $\overline{x}$: | sample mean |
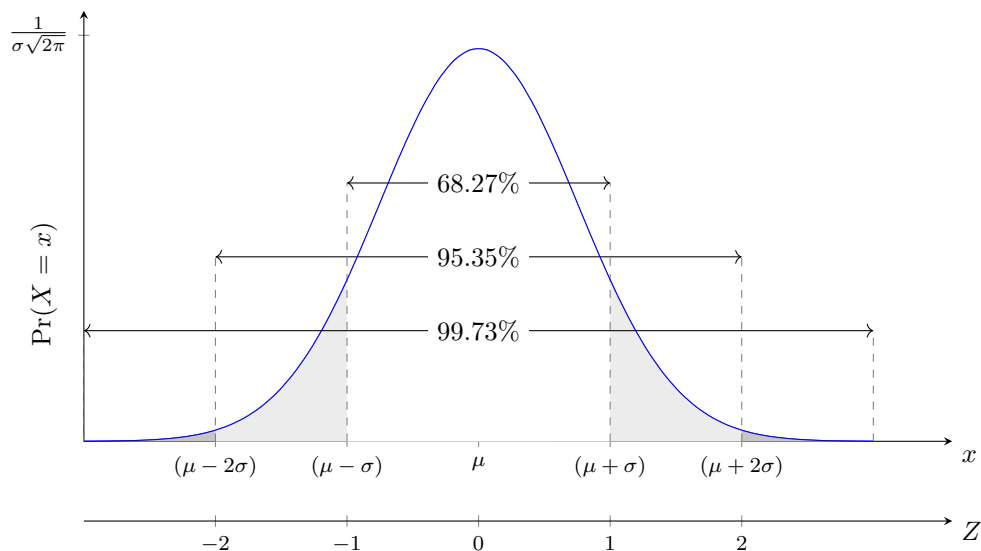> | n: | sample size |

One-tail and two-tail tests

$$p\text{-value (two-tail)} = 2 \times p\text{-value (one-tail)}$$

One tail

- μ has changed in one direction
- State "$\mathbf{H}_1 : \mu \lesseqgtr$ known population mean"

Two tail

- Direction of $\Delta\mu$ is ambiguous

- State "$\mathbf{H}_1 : \mu \neq$ known population mean"

$$p\text{-value} = \Pr(|\overline{X} - \mu| \geq |\overline{x}_0 - \mu|)$$
$$= \left(|Z| \geq \left| \frac{\overline{x}_0 - \mu}{\sigma \div \sqrt{n}} \right| \right)$$

where

   μ is the population mean under $\mathbf{H}_0$

   $\overline{x}_0$ is the observed sample mean

   σ is the population s.d.

   n is the sample size

Modulus notation for two tail

$\Pr(|\overline{X} - \mu| \geq a) \implies$ "the probability that the distance between $\overline{\mu}$ and μ is $\geq a$"

Inverse normal

> **On CAS**
>
> `invNormCdf("L", ` α`, ` $\frac{\sigma}{n^\alpha}$`, ` μ`)`

Errors

   Type I error $\mathbf{H}_0$ is rejected when it is **true**

   Type II error $\mathbf{H}_0$ is **not** rejected when it is **false**

	Actual result	
z-test	$\mathbf{H}_0$ true	$\mathbf{H}_0$ false
Reject $\mathbf{H}_0$	Type I error	Correct
Do not reject $\mathbf{H}_0$	Correct	Type II error