MATHEMATICS
SUPPORT CENTRE

Title: Differentiation 1

Target: On completion of this worksheet you should be able to differentiate functions involving powers of x.

Differentiation is the technique used to find
  the **rate of change** of a function,
  the **gradient** of a curve
  the **derivative** of a function.

First, some vocabulary :

A **variable** is a quantity that varies, as opposed to a constant, such as 3 or k.
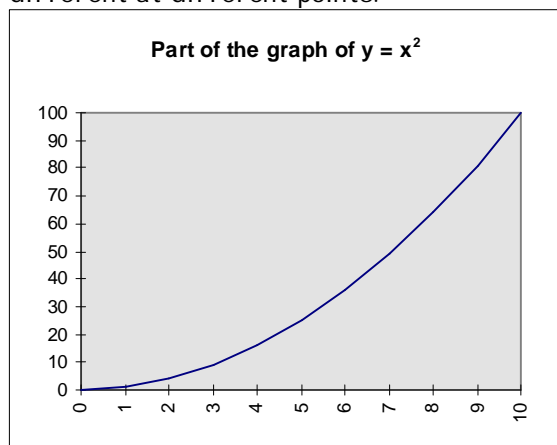
A **function** relates one variable to one or more other variables,

  e.g. $y = 3x$ is a function

y is described as being a function of x.

If $y = z^2$ we say y is a function of z ;
if $t = \sin q$ we say t is a function of q.

If you look at the graph below you will notice that the gradient (slope) of the curve is different at different points.

Part of the graph of y = x²

The gradient measures **the rate of change of** y **with respect to** x - the greater the gradient the more rapidly y is changing.

Differentiation enables us to find the gradient at any point.

What we get when we differentiate is called **the derivative** and is written as $\dfrac{dy}{dx}$.

The derivative of functions of the form

$$y = x^n \qquad \text{is} \qquad \frac{dy}{dx} = nx^{n-1}$$

eg $\quad y = x^3 \qquad\qquad \dfrac{dy}{dx} = 3x^2$

$\quad\quad y = x^4 \qquad\qquad \dfrac{dy}{dx} = 4x^3$

$\quad\quad y = x^2 \qquad\qquad \dfrac{dy}{dx} = 2x^1 = 2x$

Look again at $\quad y = x^2 \qquad \dfrac{dy}{dx} = 2x$

This tells us that the gradient of the curve at any point, x, is equal to $2x$.

  At $x = 0$ the gradient is 0 $\qquad(2 \times 0)$
  at $x = 1$ the gradient is 2 $\qquad(2 \times 1)$
  at $x = 10$ the gradient is 20 $\quad(2 \times 10)$

<u>Exercise</u> Differentiate the following and find the gradient of the curve at the given point:

1. $y = x^5 \qquad\quad x = 2$

2. $y = x^{10} \qquad\; x = 1$

3. $y = x^3 \qquad\quad x = 4$

4. $y = x^7 \qquad\quad x = 1$

5. $y = x^6 \qquad\quad x = 1$

(Answers $: 5x^4, 80; 10x^9, 10; 3x^2, 48; 7x^6, 7; 6x^5, 6)$

n can take negative values

e.g. if $y = x^{-2}$ $\qquad \dfrac{dy}{dx} = -2x^{-2-1} = -2x^{-3}$

Exercise Differentiate the following:

1. $y = x^{-1}$ $\qquad\qquad$ 2. $y = x^{-3}$

3. $y = x^{-10}$ $\qquad\qquad$ 4. $y = x^{-5}$

5. $y = x^{-7}$ $\qquad\qquad$ 6. $y = x^{-4}$

(Answers : $- x^{-2}, -3x^{-4}, -10x^{-11}, -5x^{-6}$

$-7x^{-8}, -4x^{-5}$)

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n can be a fraction

e.g. if $y = x^{\frac{1}{2}}$ $\qquad \dfrac{dy}{dx} = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}}$

Exercise Differentiate the following:

1. $y = x^{\frac{1}{3}}$ $\qquad\qquad$ 2. $y = x^{\frac{1}{5}}$

3. $y = x^{\frac{2}{3}}$ $\qquad\qquad$ 4. $y = x^{\frac{3}{4}}$

5. $y = x^{\frac{5}{4}}$ $\qquad\qquad$ 6. $y = x^{\frac{7}{5}}$

(Answers : $\frac{1}{3} x^{-\frac{2}{3}}, \frac{1}{5} x^{-\frac{4}{5}}, \frac{2}{3} x^{-\frac{1}{3}}, \frac{3}{4} x^{-\frac{1}{4}}, \frac{5}{4} x^{\frac{1}{4}}, \frac{7}{5} x^{\frac{2}{5}}$)

Two special cases:

1. If $y = a$ where a is a constant then $\dfrac{dy}{dx} = 0$

2. If $y = x$ then $\dfrac{dy}{dx} = 1$

Examples

1. $y = 6$ $\qquad\qquad \dfrac{dy}{dx} = 0$

2. $y = \frac{4}{7}$ $\qquad\qquad \dfrac{dy}{dx} = 0$

Exercise Differentiate the following:

1. $y = 4$ $\qquad$ 2. $y = x$ $\qquad$ 3. $y = 7 \cdot 894$

(Answers : 0,1,0)

Another special case:

If $y = kx^n$ where k is a constant then $\dfrac{dy}{dx} = knx^{n-1}$

e.g. if $\quad y = 2x^3$ $\qquad \dfrac{dy}{dx} = 2 \times 3x^2 = 6x^2$

$\qquad\qquad y = 4x^{\frac{1}{2}}$ $\qquad \dfrac{dy}{dx} = 4 \times \frac{1}{2} x^{-\frac{1}{2}} = 2x^{-\frac{1}{2}}$

Exercise Differentiate the following:

1. $y = 10x^5$ $\qquad$ 2. $y = 2x^4$ $\qquad$ 3. $y = 8x^{-3}$

4. $y = -4x^2$ $\qquad$ 5. $y = 3x^{-4}$ $\qquad$ 6. $y = \frac{2}{3} x^3$

(Answers : $50x^4, 8x^3, -24x^{-4}, -8x, -12x^{-5}, 2x^2$)

Examples Differentiate the following:

1. $y = 4x^2 + 3x$ $\qquad\qquad$ Differentiate term by term

$\dfrac{dy}{dx} = 4 \times 2x + 3 \times 1 = 8x + 3$

2. $y = 5 - 2x$ $\qquad\qquad \dfrac{dy}{dx} = 0 - 2 \times 1 = -2$

3. $y = x^3 + 2x^2 + 4x$

$\dfrac{dy}{dx} = 3x^2 + 2 \times 2x + 4 \times 1 = 3x^2 + 4x + 4$

Exercise Differentiate the following:

1. $y = x^5$ $\qquad$ 2. $y = x^{10}$ $\qquad$ 3. $y = x$

4. $y = 2x^3$ $\qquad$ 5. $y = 4x$ $\qquad$ 6. $y = 15$

7. $y = 2x + 4$ $\quad$ 8. $y = 3x^5 + 2x^7 + x^6$

9. $y = 9 + x$ $\quad$ 10. $y = \frac{1}{2} x^2 + 4x$

11. $y = x^2 - 3x$ $\qquad\qquad$ 12. $y = 10 - 2x^2$

(Answers : $5x^4, 10x^9, 1, 6x^2, 4, 0, 2,$

$15x^4 + 14x^6 + 6x^5, 1, x + 4, 2x - 3, -4x$)

Remember:

$\dfrac{1}{x} = x^{-1} \quad \dfrac{1}{x^2} = x^{-2} \quad \dfrac{3}{x} = 3x^{-1} \quad \dfrac{5}{x^3} = 5x^{-3}$

Exercise Differentiate the following:

1. $y = x^{-1}$ $\qquad$ 2. $y = \dfrac{1}{x}$ $\qquad$ 3. $y = x^{-3}$

4. $y = x + \dfrac{1}{x}$ $\quad$ 5. $y = \dfrac{1}{x^2}$ $\qquad$ 6. $y = \dfrac{3}{x^2}$

7. $y = x^{\frac{1}{2}}$ $\qquad$ 8. $y = x^{-\frac{1}{2}}$ $\qquad$ 9. $y = 2x^{\frac{1}{2}}$

10. $y = x^{\frac{1}{3}}$ $\quad$ 11. $y = 4x^{-\frac{1}{2}}$ $\quad$ 12. $y = 9x + x^{-\frac{1}{3}}$

13. $y = \dfrac{4}{x^2} + \dfrac{3}{x} + 5x$

14. $y = 10x^2 - 15 + x$

15. $y = 2 - 3x^2 + x^{-4}$

(Answers : $- x^{-2}, -x^{-2}, -3x^{-4}, 1 - x^{-2}, -2x^{-3}, -6x^{-3}$

$\frac{1}{2} x^{-\frac{1}{2}}, -\frac{1}{2} x^{-\frac{3}{2}}, x^{-\frac{1}{2}}, \frac{1}{3} x^{-\frac{2}{3}}, -2x^{-\frac{3}{2}}, 9 - \frac{1}{3} x^{-\frac{4}{3}}$

$-8x^{-3} - 3x^{-2} + 5, 20x + 1, -6x - 4x^{-5}$)