Cartesian components of vectors

Any vector may be expressed in Cartesian components, by using unit vectors in the directions of the coordinate axes. In this unit we describe these unit vectors in two dimensions and in three dimensions, and show how they can be used in calculations.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

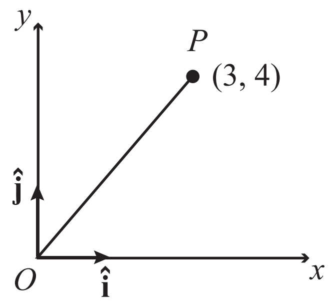
After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- identify the coordinate unit vectors in two dimensions and in three dimensions;
- determine whether a set of coordinate axes in three dimensions is labelled as a right-handed system;
- express the position vector of a point in terms of the coordinate unit vectors, and as a column vector;
- calculate the length of a position vector, and the angle between a position vector and a coordinate axis;
- write down a unit vector in the same direction as a given position vector;
- express a vector between two points in terms of the coordinate unit vectors.

Contents

1. Vectors in two dimensions

The natural way to describe the position of any point is to use Cartesian coordinates. In two dimensions, we have a diagram like this, with an x-axis and a y-axis, and an origin O. To include vectors in this diagram, we have a vector $\hat{\mathbf{i}}$ associated with the x-axis and a vector $\hat{\mathbf{j}}$ associated with the y-axis.



If we take any point in this diagram, for instance the point P with coordinates $(3, 4)$, then we can write

$$\overline{OP} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}.$$

It is important to appreciate the difference between these two expressions. The numbers $(3, 4)$ represent a set of coordinates, referring to the point P. But the expression $3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$ is a vector, the position vector $\overline{OP}$. An alternative way of writing this is as a "column vector":

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} \qquad \text{means the same as} \qquad 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}.$$

Sometimes one notation is used, and sometimes the other.

Key Point

In two dimensions, the unit vectors in the directions of the two coordinate axes are written as $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. If a point P has coordinates (x, y) then the position vector $\overline{OP}$ may be written as a combination of these unit vectors,

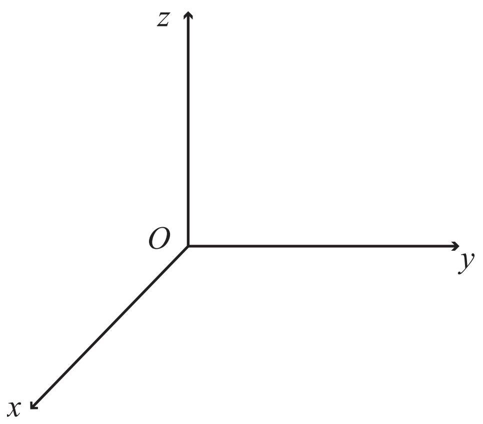
$$\overline{OP} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}},$$

or equivalently as a column vector

$$\overline{OP} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

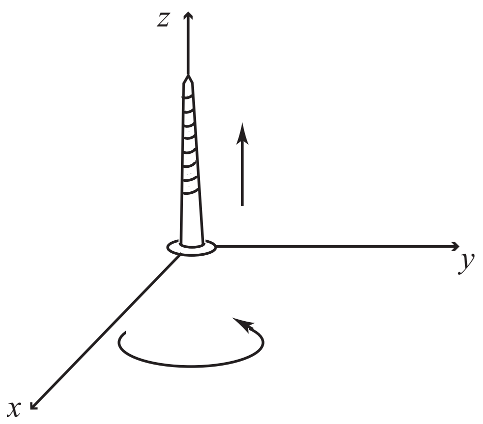
2. Vectors in three dimensions

In three dimensions we have three axes, traditionally labelled x, y and z, all at right angles to each other. Any point P can now be described by three numbers, the coordinates with respect to the three axes.



Now there might be other ways of labelling the axes. For instance we might interchange x and y, or interchange y and z. But the labelling in the diagram is a standard one, and it is called a *right-handed system*.

Imagine a right-handed screw, pointing along the z-axis. If you tighten the screw, by turning it from the positive x-axis towards the positive y-axis, then the screw will move along the z-axis. The standard system of labelling is that the direction of movement of the screw should be the positive z direction.

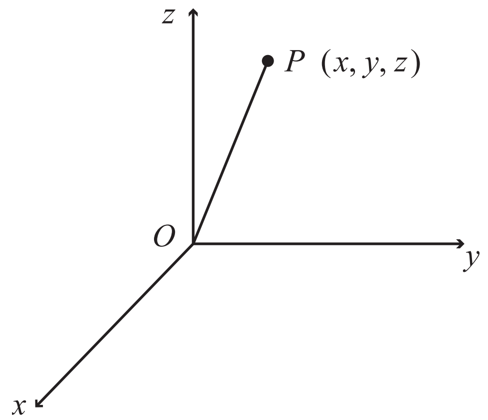


This works whichever axis we choose to start with, so long as we go round the cycle x, y, z, and then back to x again. For instance, if we start with the positive y-axis, then turn the screw towards the positive z-axis, then we'll tighten the screw in the direction of the positive x-axis.

Key Point

A right-handed system is a set of three axes, labelled so that rotating a screw from the positive x-axis towards the positive y-axis will tighten the screw in the direction of the positive z-axis.

Now let's take a point P in three-dimensional space, with coordinates (x, y, z). The position vector of the point will be the line segment $\overline{OP}$.



We can now write

$$\overline{OP} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

where $\hat{\mathbf{k}}$ is a unit vector in the direction of the z-axis. Again it is important to appreciate the difference. The numbers (x, y, z) represent a set of coordinates, referring to the point P. But the expression $x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ is a vector, the position vector $\overline{OP}$. We sometimes write this is as a column vector:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{means the same as} \qquad x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}.$$

Key Point

In three dimensions, the unit vectors in the directions of the three coordinate axes are written as $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. If a point P has coordinates (x, y, z) then the position vector $\overline{OP}$ may be written as a combination of these unit vectors,

$$\overline{OP} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}},$$

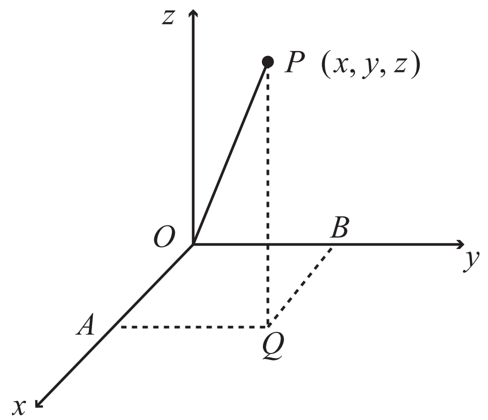
or equivalently as a column vector

$$\overline{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

3. The length of a position vector

What is the length of the position vector $\overline{OP}$?

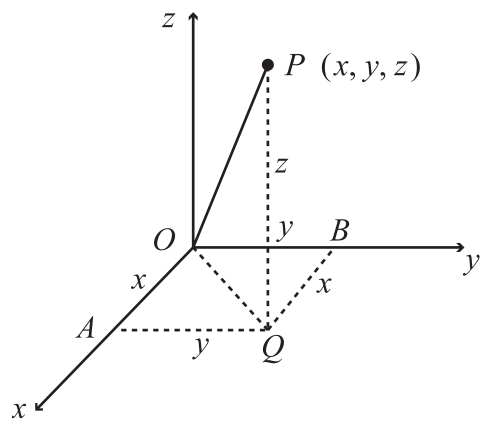
To answer this question, we start by dropping a perpendicular from P down to the (x, y)-plane. We shall call this new point Q. Then we join the point Q up to the x and y axes, again at right angles. We shall call the two new points A and B.



Now we know some of the lengths in this diagram. First, the length PQ is the height of the point P above the (x, y)-plane. So that length must be z.

The length OA is the distance along the x coordinate axis, so that length must be x. And the length BQ is the same as the length OA, so that must also be x.

In the same way, the length OB is the distance along the y coordinate axis, so that length must be y. And the length AQ is the same as the length OB, so that must also be y.



Now we join the points O and Q. Then OAQ is a right-angled triangle, and so is OBQ. So the length OQ can be found by using Pythagoras's Theorem, in either of these triangles. We obtain the formula

$$\begin{aligned} OQ &= \sqrt{OA^2 + AQ^2} \qquad \left(\text{or} \quad \sqrt{OB^2 + BQ^2}\right) \\ &= \sqrt{x^2 + y^2}. \end{aligned}$$

Now we can use the right-angled triangle OQP. If we apply Pythagoras's Theorem to this triangle, we obtain

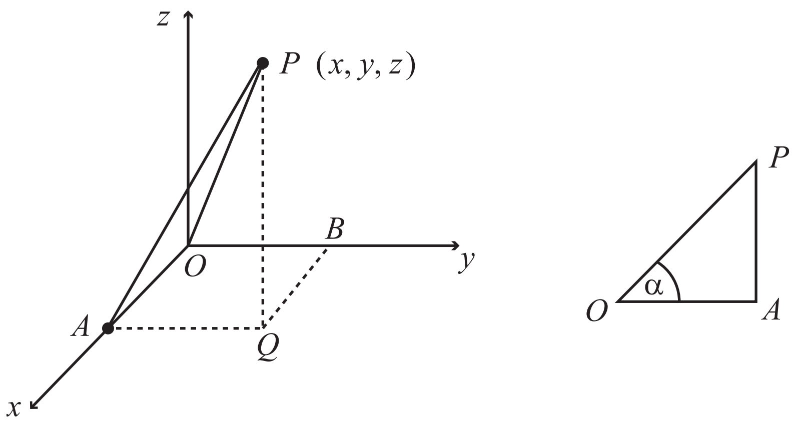
$$\begin{aligned} OP &= \sqrt{OQ^2 + QP^2} \\ &= \sqrt{\left(\sqrt{x^2+y^2}\right)^2 + z^2} \\ &= \sqrt{x^2+y^2+z^2}. \end{aligned}$$

Key Point

If P is the point with coordinates (x, y, z) then the length, or magnitude, of the position vector $\overline{OP}$ is given by the formula

$$|\overline{OP}| = OP = \sqrt{x^2+y^2+z^2}.$$

4. The angle between a position vector and an axis



Now we have found the length of the line OP: it is $\sqrt{x^2+y^2+z^2}$. And we also know the length of the line OA: it is x. But the triangle POA is a right-angled triangle, so we can write down the cosine of the angle POA. If we call this angle α, then

$$\cos\alpha = \frac{x}{\sqrt{x^2+y^2+z^2}}.$$

The quantity $\cos\alpha$ is known as a *direction cosine*, because it is the cosine of an angle which helps to specify the direction of P; α is the angle that the position vector $\overline{OP}$ makes with the x-axis.

Of course we can do the same for the y-axis and for the z-axis. We obtain

$$\cos\beta = \frac{y}{\sqrt{x^2+y^2+z^2}}$$

where β is the angle that $\overline{OP}$ makes with the y-axis, and

$$\cos\gamma = \frac{z}{\sqrt{x^2+y^2+z^2}}.$$

where γ is the angle that $\overline{OP}$ makes with the z-axis.

Key Point

The direction cosines of the point P describe the angles between the position vector $\overline{OP}$ and the three axes. If P has coordinates (x, y, z) then the direction cosines are given by

$$\cos\alpha = \frac{x}{\sqrt{x^2+y^2+z^2}}, \qquad \cos\beta = \frac{y}{\sqrt{x^2+y^2+z^2}}, \qquad \cos\gamma = \frac{z}{\sqrt{x^2+y^2+z^2}}.$$

Now we can find an interesting formula if we take the three direction cosines, square them, and add them. What is

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma?$$

Well,

$$\cos^2\alpha = \frac{x^2}{x^2+y^2+z^2}, \qquad \cos^2\beta = \frac{y^2}{x^2+y^2+z^2}, \qquad \cos^2\gamma = \frac{z^2}{x^2+y^2+z^2}$$

so that

$$\begin{aligned}\cos^2\alpha + \cos^2\beta + \cos^2\gamma &= \frac{x^2+y^2+z^2}{x^2+y^2+z^2} \\ &= 1.\end{aligned}$$

So the squares of the direction cosines, when added together, equal 1. What use could this be? In fact it tells us that the vector

$$\cos\alpha\,\hat{\mathbf{i}} + \cos\beta\,\hat{\mathbf{j}} + \cos\gamma\,\hat{\mathbf{k}}$$

is a unit vector. That is because the magnitude of this vector is the square root of the quantity $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$, which we have just seen is equal to 1. And this vector is also in the same direction as our original vector $\overline{OP}$, because the three numbers $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are in the same ratio as x, y and z. So we have found a unit vector in the direction of our original position vector $\overline{OP}$.

Key Point

The direction cosines of any point P satisfy the equation

$$\cos^2\alpha + \cos^2\beta + \cos^2 = 1.$$

A unit vector in the same direction as the position vector $\overline{OP}$ is given by the expression

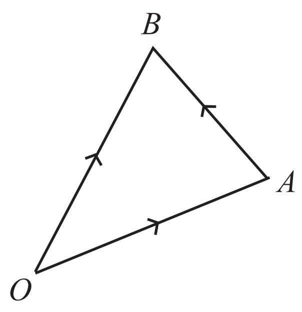
$$\cos\alpha\,\hat{\mathbf{i}} + \cos\beta\,\hat{\mathbf{j}} + \cos\gamma\,\hat{\mathbf{k}}.$$

5. An example

Suppose we have a point A with coordinates $(1, 0, 2)$ and another point B with coordinates $(2, -1, 4)$. We can then form the vector $\overline{AB}$. Now what is the magnitude of this vector, and what are its direction cosines?

We can answer these questions by writing the two position vectors $\overline{OA}$ and $\overline{OB}$ in terms of the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. We obtain

$$\begin{aligned}\overline{OA} &= \hat{\mathbf{i}} + 2\hat{\mathbf{k}}\\ \overline{OB} &= 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}.\end{aligned}$$



So

$$\begin{aligned}\overline{AB} &= \overline{AO} + \overline{OB}\\ &= \overline{OB} - \overline{OA}\\ &= (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}) - (\hat{\mathbf{i}} + 2\hat{\mathbf{k}}).\end{aligned}$$

Now when we subtract expressions involving the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ we just subtract the corresponding components separately. So

$$\begin{aligned}\overline{AB} &= (2\hat{\mathbf{i}} - \hat{\mathbf{i}}) + (-\hat{\mathbf{j}}) + (4\hat{\mathbf{k}} - 2\hat{\mathbf{k}})\\ &= \hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}.\end{aligned}$$

With this expression for the vector $\overline{AB}$, we can calculate its magnitude. It is

$$|\overline{AB}| = AB = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}.$$

We can also calculate the three direction cosines of $\overline{AB}$. They are

$$\cos\alpha = \frac{1}{\sqrt{6}}, \qquad \cos\beta = \frac{-1}{\sqrt{6}}, \qquad \cos\gamma = \frac{2}{\sqrt{6}}.$$

Exercises

1. Find the lengths of each of the following vectors

   (a) $2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

   (b) $5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$

   (c) $2\hat{\mathbf{j}} - \hat{\mathbf{k}}$

   (d) $5\hat{\mathbf{i}}$

   (e) $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$

   (f) $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

2. Find the angles giving the direction cosines of the vectors in Question 1.

3. Determine the vector $\overline{AB}$ for each of the following pairs of points

   (a) A (3,7,2) and B (9,12,5)

   (b) A (4,1,0) and B (3,4,-2)

   (c) A (9,3,-2) and B (1,3,4)

   (d) A (0,1,2) and B (-2,1,2)

   (e) A (4,3,2) and B (10,9,8)

4. For each of the vectors found in Question 3, determine a unit vector in the direction of $\overline{AB}$

Answers

1.
(a) $\sqrt{29}$ (b) $\sqrt{30}$ (c) $\sqrt{5}$ (d) 5 (e) $\sqrt{14}$ (f) $\sqrt{3}$

2.
(a) 68.2°, 42.0°, 56.1° (b) 24.1°, 111.4°, 79.5°
(c) 90°, 26.6°, 116.6° (d) 0°, 90°, 90°
(e) 36.7°, 122.3°, 105.5° (f) 54.7°, 54.7°, 54.7°

3.
(a) $6\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ (b) $-\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ (c) $-8\hat{\mathbf{i}} + 6\hat{\mathbf{k}}$ (d) $-2\hat{\mathbf{i}}$
(e) $6\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$

4.
(a) $\frac{1}{\sqrt{70}}(6\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$ (b) $\frac{1}{\sqrt{14}}(-\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$ (c) $\frac{1}{10}(-8\hat{\mathbf{i}} + 6\hat{\mathbf{k}})$
(d) $-\hat{\mathbf{i}}$ (e) $\frac{1}{\sqrt{3}}(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$